Re: derivatives and determinant

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- *From*: quasi <quasi@xxxxxxxx>
- *Date*: Tue, 14 Mar 2006 08:47:57 −0500

On Mon, 13 Mar 2006 05:53:57 −0500, quasi <quasi@xxxxxxxx> wrote:

> On Mon, 13 Mar 2006 05:23:52 EST, eugene <jane1806@xxxxxxx> wrote:
>
>> It was a typo in my first message
>> Let f(x)=(x−x_1)(x−x_2)...(x−x_n)where the numbers
>> x_1,x_2,...,x_n−pairwise distinct and a_{ii}=f''(x_i),
>> a_{ij}=(f'(x_i)−f'(x_j))/(x_i−x_j). i \neq j.
>> 1<=i,j<=n.
>>
>> Now, A mustn't be always diagonal..
>
> And with that correction, det(A)=0 now appears to be true (but I don't
> have time to look at it right now).
>
> quasi

Here is a very rough sketch of a proof strategy.

(1) What happens to A as x_i −−> x_j ?

Fix x_1 and let x_2 approach x_1. In the limit, rows 1 and 2 become equal and opposite. The same for columns 1 and 2. Hence A becomes singular as x1 −−> x2. Likewise for any pair i,j with i <> j.

(2) Simplify the entries of A to polynomials.

(3) Let f(x_1,...,x_n) = det(A).

Clearly f is a polynomial in x_1,...,x_n. Our goal is to show f=0.

(4) Let p = product (x_i − x_j) for all pair i,j with i<>j.

(5) f is a multiple of p.

From step (1), for distinct i,j, f is 0 if x_i = x_j. Hence f is a

multiple of x_i − x_j for all pairs i,j with i<>j. It follows that f is a multiple of p.

(6) f is a symmetric polynomial in x_1,...,x_n.

Again from step (1), any transposition of a pair x_i, x_j with i<>j swaps both rows i,j (with a sign change) and also columns i,j (also with a sign change). The row swap possibly negates det(A) but if so, the column swap negates it again so det(A) remains unchanged.

Since det(A) is unchanged by transpositions, it's must be unchanged for any permutation of x_1,...,x_n, hence f is symmetric.

(7) f is a multiple of p^2

By step (5), f is a multiple of (x_i − x_j) but then since f is symmetric, f must be a multiple of (x_i − x_j)^2 (otherwise there would be a sign change if x_i and x_j were swapped). Hence, f is a multiple of (x_i − x_j)^2 for all pairs i,j with i<>j. It follows that f is a multiple of p^2.

(8) Compute the degree of a_i,j with respect to x_k.

The degree of a_i,j in x_k is 0 unless k=i or k=j, in which case, the degree is n−2.

(9) The degree of f in x_k is at most 2*(n−2) = 2*n−4.

This is based on step (8) together with viewing det(A) as a signed sum of the product of generalized diagonals.

(10) Compute the degree of p^2 in x_k.

The degree of p in x_k is n−1 hence the degree of p^2 in x^k is 2*(n−1) = 2*n−2.

(11) det(A)=0

p^2 divides f but the degree of p^2 in x_k exceeds the degree of f in x_k. It follows that f=0, and hence, det(A)=0, as required.

quasi
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