Re: like a definition question about sub−algebra

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- *From*: magidin@xxxxxxxxxxxxxxxxx (Arturo Magidin)
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In article <21463759.1143224933304.JavaMail.jakarta@xxxxxxxxxxxxxxxxxxxxxx>,
eugene <jane1806@xxxxxxx> wrote:

> > What is your definition of "algebra", and what is the
> > "multiplicator
> > domain"?
>
> Thank you for reply.
> I used to the following definition of algebra:
> Algebra is a ring which is a vector space over a field F.

And do rings have to have a 1?

> > Of course, if your definition of algebra is something
> > else, then the
> > answer might be something else.
>
> Actually i was trying to solve the following problem: Let A be
> sub−algebra of R[X], generated by X^2 and X^3. Show that A isn't
> isomorphic with R[X].

Which means we are mixing up notation already, since here "A" is the
subalgebra, and in the question you asked it was the algebra... Sigh.

> Unfortunalety there wasn't a definition of the sub−algebra,generated
> by it's elements.Maybe you can deduce from the problem condition what
> was supposed by "sub−algebra of R[X], generated by X^2 and X^3".

Depending on your definition of −>RING<− (like pulling teeth...), the
answer will change.

Suppose first you do not require your rings to have a 1. Then
The subalgebra will be the smallest subspace which is closed under

products.

So take R[X], which is an algebra over R. If a subset of R[X] contains X^2, contains X^3, and is closed under products must contain all powers X^i with i>=2, since every number greater than or equal to 2 can be written as 2a+3b, with a and b nonnegative integers, not both zero. If it is a vector space it must also contain aX^i for all a in R and all i>=2. And it must contain all the sums. So it must contain all polynomials which have zero linear and zero constant term. Since this set is a linear subspace, and is closed under products, this is an algebra. It is the smallest such set, as discussed, so the subalgebra A is the conllection of all polynomials with coefficients in R which have zero constant and zero linear term.

Why is this not isomorphic to R[X]? Well, for one thing, it does not have a 1, so we are done.

Now suppose that your definition or ring −>requires<− that it have a 1. Then the subalgebra will be the smallest subspace which contains the elements given, contains 1, and is closed under products. In this case, we already know it contains all polynomials with zero linear and constant term. If we throw in 1, it is easy to verify that the subalgebra A will be the collection of all polynomials that have zero linear term.

I am guessing that this is the actual definition you have, as otherwise the result is too easy...

You want to show that it is not isomorphic to R[Y] (I'm changing notation to make the argument clearer). That is, you want to show that there is no ring isomorphism R[Y]−>A which respects scalar multiplication. Suppose that h:R[Y]−>A is an algebra homomorphism. Then h(pq)=h(p)h(q), h(p+q)=h(p)+h(q) for all p,q in R[Y], and for all p in R[y] and a in R, h(ap)=ah(p).

Now, h(1) must be either 1 or 0, since h(p) = h(1p) = h(1)h(p); if h(1)=0, then h is zero and it is not an isomorphism. If h(1)=1, then h(a)=a for all a in R, since h(a) = h(a1) = ah(1) = a. Therefore,

h(a_0 + a_1Y + ... + a_nY^n) = a_0 + a_1h(Y) + a_2[h(Y)]^2 + ... +a_n[h(Y)]^n.

Let h(Y) = b_0 + b_2X^2 + ... + b_mX^m. If b_3 is nonzero, then it should be easy to verify that no polynomial maps to X^2, so h is not surjective hence not an isomorphism. And if b_3=0, then you should be able to check that no element maps to X^3, so h is not surjective.

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"It's not denial. I'm just very selective about
what I accept as reality."
−−− Calvin ("Calvin and Hobbes")

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Arturo Magidin
magidin@xxxxxxxxxxxxxxxxx

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