Re: Carleson's proof ???

Source: http://sci.tech−archive.net/Archive/sci.math/2006−03/msg05233.html

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- *From*: rusin@xxxxxxxxxxxxxxxxxxxxx (Dave Rusin)
- *Date*: 27 Mar 2006 17:41:17 GMT

In article <270320060557405310%bruck@xxxxxxxxxxxx>,
Ronald Bruck <bruck@xxxxxxxxxxxx> wrote:

> In article <op.s62oiwp3xr5pep@thunderbird>, gooliver
> <gooliverNOSP@xxxxxxxxxx> wrote:
>
> > The theorem which Lennart Carleson has demostraded is:
> >
> > If f(x) is a real function, square−integrable then...
> >
> > Please, how to complete ???
>
> If f is in L^2(0,2pi) then
>
> f(x) = a_0/2 + \sum_{n=1}^\infty (a_n cos(nx) + b_n sin(nx))
>
> for Lebesgue almost−every x in [0,2pi], where
>
> a_n = (1/2pi) \int_0^(2pi) f(x) cos(nx) dx,
>
> b_n = (1/2pi) \int_0^(2pi) f(x) sin(nx) dx.

(Actually you need to move some constants around, e.g. remove the "2"
in the denominators of a_n and b_n .)

To complement Ron's answer let me add a little perspective.

The idea that a function can be "represented" as a Fourier series
was an ingenious contribution by Fourier himself. As I understand it
he was interested in studying temperature distributions within a
uniform disk, given a distribution of temperatures around the boundary
circle. There's a linear partial differential equation which describes
what the temperatures will be inside the disk. It's not hard to
solve that PDE when the temperature at the boundary point (cos(x),sin(x))
is something simple like cos(nx) or sin(nx), but beyond that it
becomes difficult to solve the PDE in "closed form". Fourier's

idea was simply to note that because of linearity it is sufficient to
add together the solutions from these simple cases to get solutions
to more complicated cases; that is, if C_n is the solution to the
PDE when the temperature on the boundary is given by T(x) = cos(nx),
and if S_n is the solution corresponding to T(x) = sin(nx), then
when the boundary temperature is a more complicated expression
T(x) = \sum (a_n cos(nx) + b_n sin(nx))
then the solution to the PDE is just \sum (a_n C_n + b_n S_n) .

The problem is that it's very easy to get cavalier about the use of
sums and to forget what they mean. A classic example comes from
assuming the temperature is +1 on half the circle and −1 on the
other half. (Obviously this is not completely realizable physically).
If you compute the Fourier series as Ron has indicated above, you'll
see that the a_n are zero as are the b_n for even n, but for
odd n we have b_n = (4/pi)/n . So it is tempting to just write
this boundary distribution as
\sum_{n odd} (4/pi)/n sin(n x).

But this is now an infinite sum. What does an infinite sum _mean_?
We always take this to mean the limit of the partial sums, but what
is a "limit" of functions? In this particular case, you can plot the
partials sums and see that they do indeed resemble the "+1,−1"
distribution that I started with. But there is always a little spike
near the points of discontinutiy. If you look at the n−th partial
sum f_n, the spike is narrow, and its width tends to zero as n
grows. But the height of the spike does NOT tend to zero. (This is
known as the Gibbs phenomenon.) So in what sense does the sequence
f_n of partial sums "converge" to the original f ? If you
define the "distance" between two functions to be the maximum absolute
value of their difference, then we do NOT have f_n converging to f.

But there are other ways to define a limit of functions. You could
say that f_n converges to f if \int_0^{2pi} |f_n(x) − f(x)| dx
tends to zero. Or you could say that f_n converges to f if
for each x, the sequence of real numbers f_n(x) converges to f(x) .
(Better: insist only that this happen for almost all x.)

Arguably it is the last notion of convergence which best corresponds
to our intuition, and so people wanted to prove that the Fourier
series converged to f in that sense. But that turned out to be
harder to prove than convergence in one of the other senses, such
as, in this case, L^2 convergence. That was Carleson's coup.

For those who have not heard: Carleson was awarded the Abel prize
a few days ago. This is a major award, comparable in some sense to
the Nobel prizes. More information is available at
http://www.abelprisen.no/en/

dave

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