# Re: the parabolic subgroups of SO(n,C)

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- *From*: jdolan@xxxxxxxxxxxxxxxxxxxxxxxxx (James Dolan)
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in article <1143942517.165042.313880@xxxxxxxxxxxxxxxxxxxxxxxxxxxxx>,
rupert <rupertmccallum@xxxxxxxxx> wrote:

|Does anyone know a reasonably simple description of the parabolic
|subgroups of SO(n,C)?

the way i understand it is that the incidence geometry of so(n) is based on "extension of partial isometries", in the same sense that the incidence geometry of sl(n) (aka "[n−1]−dimensional projective geometry") is based on inclusion of subspaces.

(the point of the above is that parabolic subgroups are best understood as stabilizers of configurations of "incident" geometric figures in the "incidence geometry" of which the group in question is the automorphism group; thus for example the borel subgroup (aka the smallest parabolic subgroup) of sl(3) is the stabilizer of a flag in the projective plane, a "flag" being a configuration consisting of a point and a line which are incident to each other.

given sufficient interest i might be convinced to spell out in a later post a detailed intuitive picture of the 4 basic parabolic subgroups of so(2,3) (which is equivalent to so(5) when working over the complex numbers), in terms of the "incidence geometry of extension of partial isometries" as briefly described below.)

the way it works is something like this:

we know from special relativity that a pseudo−euclidean vector space of signature (1,3) is a helpful auxiliary device for studying isometries from a euclidean vector space v of dimension 1 (aka "time") to a euclidean vector space w of dimension 3 (aka "space"), because the graph of such an isometry can be thought of as a "light−like" subspace of the pseudo−euclidean vector space v+w (aka "space−time"). this idea works pretty much just as nicely for arbitrary signature (m,n) with m<=n as it does for signature (1,3). however when "time" is multi−dimensional there can be "light−like" (aka "isotropic") subspaces of different dimensions, and the graph of even just a _partial_ isometry from time to space (defined only on some subspace of time) appears as a lower−dimensional light−like subspace of space−time, while the graph of a total isometry appears as a light−like subspace of maximal dimension.

it then turns out that any parabolic subgroup of the real lie group so(m,n=m) or so(m,n=m+1) is the joint stabilizer of some (necessarily finite) "incident" collection of partial isometries from the euclidean vector space reals^m to the euclidean vector space reals^n, where partial isometries are "incident" to each other when one is an extension of the other (to the extent permitted by the dimensions of their domains of definition).

the above can be rephrased purely in terms of the light−like subspaces given by the graphs of the partial isometries, but it's more fun to understand it in terms of the partial isometries themselves. it's also fun to work out the details that i'm sweeping under the rug here, for example how the maximal parabolic subgroups stabilizing partial isometries with domains of different dimensions correspond to the different dots in the dynkin diagrams.

the situation remains more or less the same if we work over the complex numbers instead of over the real numbers, but the concept of "signature" becomes invisible then, and thus the concept of "pseudo−euclidean vector space" merges with the concept of ordinary euclidean vector space. in fact that's why early researchers in special relativity liked to use imaginary numbers as time coordinates; it allowed them to pretend that they were still working with a euclidean vector space instead of a pseudo−euclidean one.

i haven't actually seen any published descriptions of how this all works and am only going by my own naive attempts to work it out; corrections or improvements would be welcome.

there are differing standards as to what constitutes a "reasonably simple" description, but in my opinion the kind of description given here is actually quite easy to understand, given just a very modest amount of conceptual background which unfortunately hardly ever seems to get taught.

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