Re: Question

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- *From*: "zuhair" <zaljohar@xxxxxxxxx>
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Arturo Magidin wrote:

> In article <1146146691.912732.76490@xxxxxxxxxxxxxxxxxxxxxxxxxxx>,
> zuhair <zaljohar@xxxxxxxxx> wrote:
>
> [.snip.]
>
>> I confess I have scarce mathematical knowledge, and I don't insist on
>> my beleives, I am only questioning to know the truth.
>>
>>> From what I know from Bertrand Russell's books Number x
>>> is the class
>>
>> of all similar classes containing x members (the definition seems
>> cyclic but it is not). similar classes means the existence of one to
>> one relation between their members (bijection).
>
> Try reading Halmos's "Naive Set Theory" instead.
>
> A binary relation <= on a set S is called a partial order if and only
> if it satisfies the following properties:
>
> (i) Reflexivity: a<= a for all a in S.
> (ii) Antisymmetry: if a<=b and b<=a, then a=b, for all a,b in S.
> (iii) Transitivity: for all a,b,c in S, if a<=b and b<=c, then a<=c.
>
> A partial order is a total order if for all a,b in S, either a<=b or
> b<=a.
>
> A total order on S is a well order on S if and only if for every nonempty
> subset X of S, there exists x in X such that x<=y for all y in X
> (i.e., every nonempty subset has a least element).
>
> An ordinal is a well−ordered set S such that for all a in S, the
> subset {x in S : x<a } is equal to the element a.

Some examples of ordinals: the empty set, ordered by set inclusion, is an ordinal. This is usually called 0.

The set whose only element is the empty set is an ordinal; {0} is usually called "1".

The set {emptyset, {emptyset}} is an ordinal, ordered by set inclusion. This is {0,1}, and is usually called "2".

The set {0,1,2} is an ordinal (ordered by inclusion), and usually called "3".

Continuing in this manner you obtain all the natural numbers: if you have already defined all the natural number 0 through n, then "n+1" is the ordinal {0, 1, 2, 3,..., n}, ordered by set inclusion.

In general, if a is an ordinal, ordered by set inclusion, then the set a U {a}, which contains all elments of a, plus a itself, is also an ordinal, denoted a+1. Such an ordinal is called a "successor ordinal".

omega is the union of the ordinals 0, 1, 2, 3, 4, ..., n, ...

And so on.

A cardinal is an ordinal which cannot be bijected with any ordinal strictly smaller than itself.

> So number two is the class of all doubles and number three is the class of all triples. Review " Introduction to mathematical philosophy".

Please read some MATH, not some philosophy. These definitions don't really work inside set theory, as Russell himself demonstrated. You need to go to class theory, and even there they are a bit dodgy.

> An Ordinal number is the number terms in a series, it is not the series. It is the class of all similar series.(similar has the same meaning above).
>
> A cardinal number is the number of members in a set, ie the class of all similar sets, and it is not affected by order.

Both of these notions are at odds with the standard definitions. Ordinals are specific sets, as are cardinals; they are not "quantities" or "number of elements" or any such animal. No wonder you sound like a loony.

> What is a real number ? I don't have an exact definition , but from what is writtin in this forum it seems to be a number which possess inductive properties, the kind of a number Bertrand Russells refers to as inductive number or finite number, perhaps?.

No. None of these things.

An ordinal is finite if and only if it cannot be bijected with a proper subset of itself. The "natural numbers" are the finite ordinals.

One can define an addition and a multiplication of natural numbers inductively. For any natural number n, we define

n + 0 = n
n + (a+1) = (n+a) + 1

for every natural number a.

Then one can define multiplication also inductively:

n * 0 = 0
n * (a+1) = (n*a) + n

for every natural number n.

Once you have the natural numbers, you can define the integers. One way is to consider the collection of all pairs (n,m) of natural numbers; define an equivalence relation so that (a,b) ~ (n,m) if and only if a+m = b+n. The integers are the equivalence classes under this relation.

One can show that every equivalence class contains either a pair of the form (n,0) or a pair of the form (0,n), with n a natural number. If n=0, we call this equivalence class "0". Otherwise, we denote the class of (0,n) also by "n", and the class of (n,0) by "−n".

(Intuitively, the pair (a,b) represents the solution to the equation "a+x=b"; define addition and multiplication accordingly)

It is then an easy exercise to show that the map that sends the natural number n to the class of (n,0) respects addition and multiplication and order, so that we can view the natural numbers as contained in the integers, and this abuse of notation is justified.

Once we have the integers, you can define the rationals. Consider the set of all pairs (a,b) of INTEGERS, with b nonzero. Define the equivalence relation (a,b) ~ (x,y) if and only if ay=bx. (Intuitively, the pair (a,b) represents the solution to the equation ax=b).

A "rational number" is an equivalence class under this. We denote the equivalence class of (a,b) by "a/b"; each equivalence class contains one and only one pair (a,b) such that gcd(a,b)=1, and then "a/b" is the "expression in least terms". Define addition by (a,b) + (x,y) = (ay+bx,xy), and multiplication by (a,b)*(x,y)=(ax,by).

It is then also an easy exercise to show that the rationals contain a copy of the integers, namely the integer n corresponds to the rational n/1 (or to the class of (n,1)).

Once you have the rationals, the reals can be constructed in any number of ways.

A "Dedekind Cut" of the rationals is a partition of the rationals into two sets, (A,B), such that:

(i) A U B is all the rationals.
(ii) A/\B is empty.
(iii) for each a in A and b in B, a<b.
(iv) A and B are each nonempty.

These partitions come in three flavors: either A has a largest element (in which case B has no smallest element); or B has a smallest element (in which case A has no largest element); or A has no largest and B has no smallest element.

For example, the partition A = {x : x <=0}, B = {x : x>0} is of the first type.

The partition A = {x : x<1}, B = {x: x>=1} is of the second kind.

The partition that has B = { x: x>0 and x^2 >= 2}, and has A =Q−B is of the third kind.

The real numbers can be defined in terms of equivalence classes of these Dedekind cuts. Intuitively, in either case 1 or case 2, the cut represents a rational; in case 3, it represents an irrational.

Read "Continuity and irrational number" by Richard Dedekind; in "Essays on the Theory of Numbers" by Richard Dedekind, translated by Wooster Woodruff Beman. Dover Publications, Inc.

Alternatively, one can define a "distance" of rational numbers by the usual means: the distance between a/b and c/d is |ad−bc|/bd, where |ad−bc| is the absolute value of ad−bc.

> A sequence of rationals is function from the natural numbers to the rationals.
>
> We say a sequence (a_0, a_1,...) (usually denoted {a_i}) converges to the rational number Q if and only for every N>0 there exists M>0 such that if n>M, then |a_n−Q|<1/N.
>
> We say a sequence (a_0, a_1, ...) is a "Cauchy sequence" if and only if for every N>0 there exists M>0 such that if n,m>M, then |a_n−a_m|<1/N.
>
> It is easy to verify that if a sequence converges to some rational, then it is Cauchy, though the converse does not hold.
>
> We can define an equivalence relation among sequences by saying that the sequence {a_i} and the sequence {b_i} are "equivalent" if and only if the sequence {a_i−b_i} is a Cauchy sequence.
>
> It is an easy exercise to show that if {a_i} converges to q and {a_i} is equivalent to {b_i}, then {b_i} also converges to q. And if {a_i} is Cauchy and {b_i} is equivalent to {a_i}, then {b_i} is also Cauchy.
>
> The "real numbers" can be defined to be the set of all equivalence classes of Cauchy sequences of rationals. Those that converge correspond to the rationals; those that do not converge to a rational correspond to the irrationals.
>
> It is this latter definition that gives rise to the numerical representation. A decimal expansion
>
> N.d1d2d3....
>
> with N an integer, di an integer between 0 and 9, is short hand for the sequence
>
> (N, N+d1/10, N+(d1/10)+(d2/100), ..., N + (d1/10) + (d2/100) + ... + (dn/10^n),...)
>
> which can easily be verified is a Cauchy sequence; so the decimal expansion represents the EQUIVALENCE CLASS of cauchy sequences corresponding to this sequence. It is again a trivial exercise to show, for example, that the Cauchy sequence represented by 1.000000... (which is the constant sequence (1,1,1,1,...)) and the Cauchy sequence represented by 0.9999.... (which is the sequence (9/10, 99/100, 999/1000, ...)) are equivalent Cauchy sequence. Therefore, a fortiori, they represent the same "real number".

Tell me what is that trivial exercise.

Zuhair

> I heard that division is not defined for ordinals , but I don't know why? Division of transfinite cardinals by finites is defined,

No, it is not defined. At least, not in STANDARD cardinal theory. You can define anything you want, naturally. But for example, the USUAL meaning of division is that "a/b" represents the UNIQUE element c such that b*c = a. This does not work for cardinals, except in the very limited situation in which a and b are both finite, and b divides a (in the usual sense of natural numbers).

Do read some set theory, man.

> why division and subtraction is not defined for ordinals?

Because division is defined in terms of the inverses for multiplication, and subtraction is defined in terms of the additive inverses. Neither of them exist for ordinals nor for cardinals.

Thanks .

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"It's not denial. I'm just very selective about
what I accept as reality."
−−− Calvin ("Calvin and Hobbes")
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Arturo Magidin
magidin@xxxxxxxxxxxxxxxxx

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