Re: a proof for consideration

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* *Date*: 11 Sep 2006 21:59:44 −0700

I finally got my hands on a copy of Steinberg's paper, and the key line of reasoning I use to make the proof that P is not equal to NP is not addressed in the paper. The focus of that paper is on finding local structures that would determine 3−colorability, while I demonstrate that such an approach cannot be satisfactory. Furthermore, I show that reducing the problem to one that is local in scope necessarily requires exponential time as well.

As far as I can tell, based on the Steinberg paper (which, admittedly, is 13 years old), the approach I used is novel. I haven't come across anything more recent that would suggest otherwise, either.

Matt Zellman wrote:

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>> Proginoskes wrote:
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>>>>> matt.zellman@xxxxxxxxx wrote:
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>>>>>> so a couple of weeks ago, I posted here asking for help with a paper. I decided to go ahead and put what I have out here so people can

look it over. I'll reiterate some of the concepts I went over in the previous thread, and then see if my reasoning is sound.

Recently, I got a few free moments, and I looked up Richard Steinberg's paper "The Three−Color Problem" (to update and fix errors on my page about Steinberg's Conjecture), and I found that some of Zellman's concepts showed up there as well.

1. k−chromatic Edge Replacement Subgraphs (^kERSs)

a ^kERS is a k−chromatic graph that contains at least one pair of nonadjacent vertices for which no valid k−colorings exist when they are colored the same color.

A couple of Russian
mathematicians, V.A.
Aksionov and L.S.
Mel'nikov,
called ^kERS's
"quasi−edges" in a 1978
paper ("Essay on the theme:
the
three−color problem",
Combinatorics, Colloquia
Mathematica Societatis
Janos Bolyai 18, 23−34).

Would it be more appropriate for me to
switch to this term, since it
predates mine? I think it is much clearer to
include the chromatic
number in the designation, because such
subgraphs only really work when
colored with a set number of colors.

Sticking with "quasi−edges" would cause less confusion.
However, if
calling them "k−quasi−edges" would be a nice compromise.

Sounds good to me. "k−quasi−edges" it is.

The ^kERS
as a whole
functions in
exactly the
same way
as a single
edge, and a
k−chromatic
graph can
be
transformed
G=>G' by
replacing an
edge with a
^kERS, a
process I

have called "expansion by edge−replacement."

The reverse process, replacing a ^kERS with an edge, I have termed "reduction by edge−replacement."

2. Boundary Points

Every graph has at least one set of vertices (of a size greater than or equal to the chromatic number k of the graph) for which no valid k−colorings exist when the vertices in the set are colored with less than k colors. Any such set of vertices is a set of "boundary points."

3. Basic k−chromatic Graphs

A primary basic

k−chromatic graph is constructed by taking a basic (k−1)−chromatic graph and adding one vertex, which is then connected to an entire set of boundary points with edges or ^kERSs. A secondary basic k−chromatic graph is an expansion of a primary basic k−chromatic graph by edge−replacement. The basic 1−chromatic graph is a single vertex.

Every graph with chromatic number k contains at least one basic k−chromatic graph as a subgraph, and no basic (k+1) chromatic graphs as subgraphs.

> > > > This sounds a lot like a
> > > > problem that Bjarne Toft
> > > > raised in 1985:
> > > >
> > > > PROBLEM. Suppose G is a
> > > > (k+1)−colorable graph
> > > > which does not contain
> > > > K_(k+1). Does it follow that
> > > > there are two vertices x and
> > > > y and two
> > > > k−colorable subgraphs G1
> > > > and G2, each containing x
> > > > and y, such that:
> > > > (1) in any k−coloring of G1,
> > > > x and y receive different
> > > > colors, and
> > > > (2) in any k−coloring of G2,
> > > > x and y receive the same
> > > > color.
> > > >
> > > > The converse is true for any
> > > > k. This problem is true for
> > > > k=2 but has
> > > > been proven to be false if k
> > > > >= 6. It is open for k=3,
> > > > AFAIK.
>
> > > Basically, the question is, "is there any
> > > combination of a ^kERS and an
> > > 'anti−^kERS' that would force k+1 colors,
> > > and does not contain a basic
> > > k−chromatic graph?"?
> > >
> > > right?
>
> > Yes. Although here, all "boundary sets" would have size two.
>
> interesting. I think I can prove that the answer to this question is
> "no" for k=3.

should I go ahead and attempt this proof? or is it even necessary?

The scope of the three−color problem is bounded only by the size of the graph. That is, consider the graph below (Figure A):

```
O−−−O−−−O−−−O−−−O
/|\ / \ / \ / \ /|
O−O O O O
O |
\|/ \ / \ / \ / \|
O−−−O−−−O−−−O−−−O
```

This graph is 4−chromatic, as are all the graphs with the same end regions and different lengths of the same pattern in the middle. I could change something anywhere on the graph to make the chromatic number 3. For example, I could delete the leftmost vertex, I could add a vertex in the middle of the edge at the right end, or I could change any of the middle vertices to other configurations.

To put it succinctly, in order to determine the colorability of the graph, I am required to make an exhaustive analysis. No local analysis would guarantee that I find all the structures that determine the colorability of the graph.

Suppose, however, that using edge−replacement, we could reduce the graph to some configuration that could be analyzed locally. If we replace all the ^3ERSs with edges until there are no ^3ERSs left, we will be left with a graph that can be analyzed locally, in deterministic polynomial time. We simply have to go through the graph, and for each vertex, we look for all the edges connected to that vertex, and take note of which vertex is on the other end of each edge. Then we look for all the edges between those vertices, and build a subgraph from them. We can try to color this subgraph with two colors. If we can, then the graph may still be 3−colorable.

If we can't, then the graph is not 3−colorable.

It certainly seems like a reasonable course of action, but it turns out that even identifying ^3ERSs in a graph is necessarily exponential over the inputs. Go back to the example graph shown above. If we take it back to the basic 3−chromatic graph it is constructed from, we are left with the graph below (Figure B):

```
O−−−O−−−O−−−O−−−O
/ \ / \ / \ / \ /|
O O O O O
|
\ / \ / \ / \ / \|
O−−−O−−−O−−−O−−−O
```

It is an example of a graph in which a single ^3ERS has been

iterated 4
times from
a simple
triangle.
The ^3ERS
by itself
looks like
this
(Figure C):

```
1@−−−O2
\ /|
3O |
/ \|
4@−−−O5
```

(where the
@s signify
the
endpoints of
the edge
that is being
replaced,
and the
vertices are
numbered
from 1−5 to
help us out
later)

Aksionov and Mel'nikov
call the smaller graph a
"building block".
Richard Steinberg's paper
goes into more detail.

Where would be the best place to get ahold
of these papers? What
exactly am I looking for?

The Steinberg paper is in a book called _Quo Vadis, Graph
Theory?_,
which should be in your local university library (call number
QA166 .Q6
1993). (This book is about 400 pages long, but Steinberg's
article only
consists of pages 211−248.) I would think it's too old to be
available

electronically. Steinberg _might_ still have preprints or
reprints, but
I doubt it. (His webpage is
http://www.jbs.cam.ac.uk/people/faculty/steinbergr.html).

I will check this out the next chance I get.

I finally located a copy and am waiting for it to come in. There isn't a library in the area that has it.

In each
even
iteration of
this ^3ERS,
an edge can
be
constructed
between the
initial
vertex 1,
and vertex 5
of the
second/fourth/sixth...
iteration,
without
changing
the
colorability
of
the graph.
Alternatively,
an edge can
be
constructed
between the
initial
vertex 4 and
vertex 2 of
that
iteration.
For odd
iterations,
they switch:
initial
vertex 1 can
be

connected to vertex 2 of the third/fifth/seventh... iteration, or initial vertex 4 can be connected to vertex 5 of that iteration.

Suppose that for each iteration, we construct one of these two edges. Such a construction across iterations creates a ^3ERS that cannot be reduced to another ^3ERS, but only straight to an edge. Each iteration adds 3 vertices and 7 edges to the graph. The extra edge we add makes it 3 and 8. We can represent which edge we have constructed at each iteration by simply making an ordered list: 14411144444... This particular example is equivalent to the graph represented by 41144411111, but not to the graph represented by 11411144444 or any other graph in the set. There are therefore at least 2^(i−2) unique possible ^3ERSs for each iteration i (i>1). Relating this to the size of the input (suppose our input is just the list of edges), the number of necessary tests for unique ^3ERSs for a graph with E edges must necessarily exceed (since our starting set of ^3ERSs is severely limited, as are the rules for construction):

2^(E/8−2) for E>16

Since without edge−replacement, the scope of the problem is unbounded, and therefore requires an exhaustive search, and with edge replacement, it requires a number of tests that is at least exponential over the size of the input, we are forced to conclude that the 3−coloring problem cannot be solved deterministically in polynomial time.

And, as a direct result, P is not equal to NP.

Is there anything I've missed?

Is this part of the proof new? or is it also
covered in Steinberg or
Askionov and Mel'nikov?

This last part isn't, but Steinberg usually only summarizes results.
The reference list has 128 papers on it.

−−− Christopher Heckman

Oh, fun, a scavenger hunt! ;−)

~Matt Zellman

As a footnote, I realize I never actually stated the purpose of the exhaustive search in the proof, which leaves a rather significant disconnect for people that don't make the connection automatically. What we are searching for are the new points that were added in the construction of a basic 4−chromatic graph. In a sense, we are seeing whether the vertices directly connected to some particular vertex are a set of boundary points of a basic 3−chromatic graph (or really, any 3−chromatic graph, which if the conjecture about basic 3−chromatic graphs being inherent in all graphs of chromatic number 3 is true, amounts to the same thing). The scope of the search is the number of vertices removed from the initial vertex that we have to examine.

If it is necessary, I can prove that this is the most efficient algorithm possible (other than guess−and−check, which may actually be faster−−as is the case for 2−coloring), though I have a hunch that this statement may be equivalent to−−or at least follow quickly from−−what was proved by Razborov and Rudich regarding "natural proofs."

does Razborov and Rudich's proof actually imply this, or is it wishful thinking on my part?

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