Re: Galois theory problem: if a^3+a+1=0, does Q(a) contain i=sqrt(−1)?

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Snis Pilbor wrote:

> Hello, here is yet another qual practice problem I wasn't able to solve.
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> "Suppose a is a complex number with a^3+a+1=0. Determine whether or not sqrt(−1) lies in Q(a)."
>
> Part of the problem is that I'm unable to factor the polynomial, even over C. I'm sure there is supposed to be a solution which doesn't involve factoring the polynomial.
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> The main idea I was trying was to show that [Q(a):Q]=3. If this were true, sqrt(−1) could not lie in Q(a) because the min polynomial of sqrt(−1) is degree 2 and 2 does not divide 3. The polynomial has two real roots and one complex root. If it had one real and two complex, its Galois group would have S_3, but the converse is false (I think?), and I don't know how to show that the Galois group is not S_3.
>
> Or heck, maybe the Galois group *is* S_3. I don't know how to tell. I know that since the polynomial is cubic, there are some fancy tests using the discriminant to mechanically find the Galois group, but these are not the kind of things one memorizes by rote, and I doubt they'd give full credit for such a solution anyway since that material isn't among the "assumed knowledge" for the qual.
>
> Thanks a ton for any help you can give =)

First of all, if this polynomial is reducible, then it has an integer root, which must be plus or minus 1. Since neither of these work, the polynomial if irreducible over Q. Hence adjoining a root gives a field extension whose degree is the same as the polynomial, namely 3. But i is of degree 2 over Q, so [Q(i):Q] = 2. But an extension of degree 3 cannot contain an extension of degree 2, because when you have a tower of field extensions, the degrees multiply. The Galois group is completely irrelevant. Some people call anything to do with field

extensions Galois theory, but some don't. This is pretty basic stuff.
Some review seems to be in order.

Regards,
Achava

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