# Exotic functions (elementary)

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I've recently made a couple of posts to another
group (not usenet) that I thought might be of
interest to some people in sci.math, and so
I've merged those posts together to form this one.

http://mathforum.org/kb/thread.jspa?threadID=1463266

Incidentally, neither of the properties being considered
(unbounded in every interval, dense graph) can occur for
a Baire one function. However, the first example I give
is a Baire two function, and I'm pretty sure it's possible
(use condensation of singularities?) to obtain a Baire two
function whose graph is dense in the plane.

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1. A FUNCTION UNBOUNDED IN EVERY INTERVAL

I had a little free time, so I thought I'd
discuss a function, related to the ruler function

<http://mathforum.org/kb/thread.jspa?messageID=4594375>,

whose behavior illustrates a possibility that some
might consider impossible for a function to have.

We define f from the reals to the reals as follows.

f(x) = 0 if x is irrational or x=0.

f(p/q) = q if x is rational and x = p/q in lowest terms and x has the same sign as p (i.e. q > 0).

Then f has the property of being unbounded in every interval. To see this, consider an arbitrary interval J. Then, for each integer n > 0 (no matter how large), there exist infinitely many reduced−form fractions p/q in J such that p > n. This is easy to see if you consider "grids" on the number line formed by marking the points

...., −3/q, −2/q, −1/q, 0, 1/q, 2/q, 3/q, ...,

first for q = M+1, then for q = M+2, then for q = M+3, and so on. If the interval J is very short, the first few of these grids might not produce any points in J, but once q gets large enough, the grids will start producing points in J. Of course, only those p/q's where p and q are relatively prime will matter, but this is easily taken care of by just considering those integers q > M such that q is a prime number.

Note that the value of f is bounded at each point. [All I'm saying is that for each real number r, |f(r)| < oo, which is a no−brainer.] However, for each point r, the limiting behavior of f at x=r is "unbounded". By this, I mean that f is unbounded in every neighborhood of x=r.

Incidentally, f is unbounded in only one way, namely from above. However, if we replace "f(p/q) = q" with "f(p/q) = [(−1)^p] * q", then I believe we'll get a function whose graph is unbounded, both from above and from below, in every interval.

As badly behaved as this function is (it's worse than being discontinuous at each point, because a function can be discontinuous at each point and still be bounded on the entire real line), its graph is relatively sparse in the plane. For example, the graph of this function is certainly a subset of the union of all horizontal lines having an integer y−intercept, and even this larger set misses a lot of regions with positive area in the plane (e.g. the interior of the circle with center (1, 1/2) of radius 1/2).

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## 2. A FUNCTION WHOSE GRAPH IS DENSE IN THE PLANE

Now I'll show that we can define a function g, much worse than this, so that the graph of g is dense in the plane. This means that the interior of *every* circle in the plane will contain points belonging to the graph of g.

Let D_1, D_2, D_3, ... be an infinite sequence of sets of nonzero real numbers such that each pair of the sets has empty intersection and each of the sets is a dense subset of the real numbers. See below for several ways to obtain such a sequence of sets.

Write the rational numbers as a sequence r_1, r_2, r_3, ... We can do this because the rational numbers form a countable set. Indeed, there are many explicit methods for carrying this out −− google phrases such as "rationals are countable", "list the rationals", "countability of the rationals", etc.

For each n = 1, 2, 3, ..., let L_n be the horizontal "line" L_n = { (x, r_n) : x belongs to D_n }.

That is, L_n is the set of all points in the plane whose x−coordinates are numbers chosen from D_n and whose y−coordinates are the (fixed) number r_n.

The function g, and here I'm identifying the function with its graph, consists of all the points in any of the sets L_n, along with every point (if any) of the form (x',0), where x' is a real number that doesn't belong to any of the sets D_1, D_2, D_3, ...

g is function −− It is easy to see that g satisfies the vertical line test by using the fact that the sets D_1, D_2, ... are pairwise disjoint and the fact that none of these sets contains 0. Also, the domain of g is the set of all real numbers. This is because each real number is either in one of the "D−sets" (in which case g contains an ordered pair whose x−coordinate is that real number) or it isn't in any of the "D−sets" (in which case the value of g at that real number is 0).

g is dense in the plane −− It suffices to show that the (possibly) smaller set consisting of all the horizontal "lines" L_1, L_2, ... is dense in the plane. To see this, we need to show that each rectangle [a,b] x [c,d] in the plane contains in its interior at least one point belonging to these "L−sets". Because the rational numbers are dense, there exists a rational number, call it r_k (i.e. let k be the index for one such rational number in our listing of the rational numbers as r_1, r_2, ...), such that c < r_k < d. Because the set D_k is dense, there exists a number s in D_k such that a < s < b. Then (s, r_k) belongs to L_k, and hence to the union of the horizontal "lines", and (s, r_k) lies in the interior of the rectangle [a,b] x [c,d].

HOW TO OBTAIN THE SETS D_1, D_2, D_3, ...

One way is to let D_1 be the set of all nonzero rational numbers whose reduced−fraction form has a denominator that is some power of 2, D_2 be the set of all nonzero rational numbers whose reduced−fraction form has a denominator that is some power of 3, ..., D_n be the set of all nonzero rational numbers whose reduced−fraction form has a denominator that is some power of the n'th prime number, ...

A 2'nd way is to let D_1 be the set of all nonzero rational numbers whose reduced−fraction form has a prime denominator, D_2 be the set of all nonzero rational numbers whose reduced− fraction form has a denominator that is a product of two prime numbers, ..., D_n be the set of all nonzero rational numbers whose reduced−fraction form has a denominator that is a product of n prime numbers, ...

A 3'rd way is to let
D_1 = { t + sqrt(2): t is a rational number}
D_2 = { t + sqrt(3): t is a rational number}
..
..
..
D_n = { t + sqrt(p_n): t is a rational number},
where p_n is the n'th prime number,
..
..
..

A 4'th way is to let
D_1 = { t*sqrt(2): t is a nonzero rational number}
D_2 = { t*sqrt(3): t is a nonzero rational number}

..
..
..
D_n = { t*sqrt(p_n): t is a nonzero rational number},
where p_n is the n'th prime number,
..
..
..

The 4'th way is probably the easiest to verify the properties for. To show D_n is dense, let a,b be two real numbers with a < b. Choose t' to be a nonzero rational number such that t' lies between a / sqrt(p_n) and b / sqrt(p_n). [We can do this because the rational numbers, and hence the nonzero rational numbers as well, are dense in the reals.] Then t'*sqrt(p_n) lies between a and b. To show that the sets are pairwise disjoint, suppose t*sqrt(p) belongs to one of the sets and t'*sqrt(p') belongs to another one of the sets. Now it might be the case that t = t', but since the two numbers are taken from different "D−sets", we must have p not equal to p'. Could these two numbers be equal? No, because if t*sqrt(p) = t'*sqrt(p'), we'd have t/t' = sqrt(p'/p), a contradiction, since t/t' is a rational number and sqrt(p'/p) is an irrational number (because the ratio of two different prime numbers can't be a perfect square).

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