Re: Ordinal numbers and rings

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In article <1163171016.367349.35860@xxxxxxxxxxxxxxxxxxxxxxxxxxxx>,
Jose Capco <cliomseerg@xxxxxxxxxxxxxxxxxxxxxxxxx> wrote:

> Arturo Magidin wrote:
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>> Depends on the family! You should probably post the entire thing. I suspect that this is much more a local issue than some sort of underlying principle you seem to be groping for.
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> All right, allow me to post it. It's only a few sentence and I suppose it's not difficult to understand if one knows how to deal with the ordinals here.
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> This is a paper by Edgar Enochs on "Totally Integrally Closed Rings", Proceedings of the American Mathematical Society, Vol. 19, No. 3 (Jun., 1968), pp. 701−706
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> I will quote a statement on the first paragaraph of the proof on page 705:
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> −−−−−−−−−−−−−−−
> suppose A is a subring of B, B is a limit ordinal number (−− maybe this is a misprint, because I have this as an OCR scan of the original text, so maybe he meant beta instead of B−−−−), and (A_alpha) is a family of subrings of B indexed by alpha<beta such that:

That sounds exactly right: the text should likely be

"Suppose A is a subring of B, and beta is a limit ordinal number, and {A_alpha} is a faimly of subrings of B indexed by alpha < b such that..."

A_0 = A
A_(alpha+1) is a tight extension of A for all alpha<beta (−−− tight extension means that nonzero ideals of A_(alpha+1) if intersected with A are nonzero ideals −−−)

A_gamma = \bigcup A_alpha for alpha<gamma whenever gamma<beta is a limit ordinal.

Then it is easy to check that \bigcup A_alpha for alpha<beta is a tight extension of A.
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The text before the one I just quoted isnt relevant, at most it is assumed that B is a tight extension of A.. but I don't think being a tight extension has anything to do with the construction that is being made...

I just want to understand the construction.. and the last sentence which apparently assumed that \bigcup A_alpha is a ring... do I need to get myself of a book on modern set theory to understand this?

I don't think so.

Okay; you have a family of rings, A_{alpha}, indexed by an ordinal beta which happens to be a limit ordinal.

You want to show that the union of the A_{alpha} is a ring. The key, really, is to be able to show that the sum and product of two elements of the union is well defined. This requires showing the following: if a,a' are in \/A_{alpha}, then both a+a' and aa' are defined. (The other properties will follow from this). The typical way to do this when you have a union of algebraic structures is the following:

given a,a' in \/A_{alpha}, there exist alpha and alpha' such that a is in A_{alpha} and a' is in A_{alpha'}. We want to find some index i such that A_{alpha} and A_{alpha'} are both contained in A_i, and then we can define a+a' and aa' simply as "whatever a+a' and aa' are in A_i".

I suspect the author wants to use the fact that although for successor ordinals A_{alpha} and A_{alpha'} need not be related (other than being extensions of A), if you can find a limit ordinal gamma such that alpha,alpha' <= gamma < beta, then A_{alpha} and A_{alpha'} are both contained in A_{gamma}, and you can go from there.

That said, however, I think the author is wrong (or at least, that his assertion is incomplete). For example, let beta = omega = w, the natural number, which is a limit ordinal. Let A = Q, B=algebraic closure of Q, and define A_n for natural number n as follows:

A_0 = Q

A_n = Q(sqrt(p_n)), where p_n is the n−th prime.

Then A_n is a ring extension of A for every n, and it is a tight extension, since the only nonzero ideal of A_n is A_n itself, which intersect A in a nonzero ideal (namely A).

This means that A_{n+1} is a tight extension of A for every ordinal n<w, and by vacuity A_gamma is the union of all A_alpha with alpha<gamma, when gamma is a limit ordinal strictly less than omega.

However, the union of the A_n is not a ring, since for example sqrt(2)+sqrt(3) is not defined and is not even in any of the A_n.

The problem is that if you look at ordinals for which there is no limit ordinal strictly between them and beta, you have no guarantee that you can fit them both inside a single ring in the family...

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"It's not denial. I'm just very selective about
what I accept as reality."
−−− Calvin ("Calvin and Hobbes" by Bill Watterson)
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