Re: Moebius Band is not homeomorphic with a Torus

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- *From*: lrudolph@xxxxxxxxx (Lee Rudolph)
- *Date*: 19 Nov 2006 12:15:18 −0500

David Marcus <DavidMarcus@xxxxxxxxxxxxxx> writes:

> Narasimham wrote:

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>> David Marcus wrote:

....

>>> However, two surfaces can be homotopic without being homeomorphic.
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>> You have to give some more examples (of course other than the present one) and its logic, if there is something that can be brought in here.
>
> Do you mean you want an example of surfaces that are homotopic, but not homeomorphic?

A nice example, which ought to (but probably won't) clear up N.'s confusion, is the following. Take that inner tube he had a while ago, and rip a patch off it. This physical object, which many people would be content to abstract to S, a torus with an open disc removed−−i.e., an orientable surface of genus 1 with a single boundary component−−*does* have thickness, and so it would appear that N. would like to abstract it differently, as the "square tube" (if I understand his term correctly) of that surface, in other words, as a trivial I−bundle over S, which is a 3−dimensional solid with piecewise smooth boundary comprising three parts−−a (very narrow) annulus (which is the restriction of the I−bundle to the boundary of S) and two copies of S (the "inside" and "outside" surfaces of the inner tube). Of course homeomorphisms don't care about corners, so topologically this solid is a "handlebody of genus 2", bounded by a surface F of genus 2 (with empty boundary); the residuum of the corners is a decomposition (what Gabai calls a "suturing") of

F into an annulus and the two (as it happens, homeomorphic) components
of its complement.

Now start again, this time from a flat disk from the interior of which
two smaller open disks with disjoint closures have been removed. A
physical counterpart of this might be a mask with two eye−holes (and
no mouth or nostrils), or a pair of underpants (with no fly). Again,
many of us would be content to abstract the mask or pants to S',
the original orientable surface of genus 0 with three boundary components;
but N., presumably, would once again take the flat tube. This time
we *again* get a 3−dimensinal handlebody of genus 2, but now the
sutures are different: there are 3 of them.

The process of replacing S or S' with a tribial I−bundle over it does not
change the homotopy type, and indeed creates two homeomorphic total spaces;
yet (of course!) S is not homeomorphic to S'.

Put differently, cancellation doesn't work for homeomorphism types:
you can have A x X = B x X with X non−empty (even, like I, contractible)
and yet not have A = B.

Lee Rudolph
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