Re: f and f' square integrable

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- *From*: David C. Ullrich <ullrich@xxxxxxxxxxxxxxxx>
- *Date*: Mon, 01 Jan 2007 15:15:32 −0600

On Sun, 31 Dec 2006 14:32:45 −0800, The World Wide Wade
<waderameyxiii@xxxxxxxxxxxxxxxxxxxx> wrote:

> In article <NDHlh.1114208$084.1066831@attbi_s22>,
> Stephen Montgomery−Smith <stephen@xxxxxxxxxxxxxxxxx> wrote:
>
>> Fedor wrote:
>>
>>> Hi all,
>>>
>>> suppose that f:R^n −−>R is a smooth function such that f^2
>>> and (f '
>>>)^2 are integrable over R^n. Is it true that f(x) tends towards
>>> 0 when
>>> |x| tends towards infinity ? It is easy for n=1 but is it true for
>>> the
>>> general case ?
>>>
>>> Regards,
>>> fedor
>>
>> I think not.
>>
>> First consider a function like f(x)=log(1/|x|)^a for 0<a<1/2. Check
>> that f and f' are square integrable in R^2.
>>
>> Next consider sum a_n f(x−(2n,0)) where a_n is square summable.
>
> If f(x) = [log(1/|x|)]^a for x near 0, then f is not smooth at 0.
> Perhaps you can play around with this to arrive at an example,
> although the obvious things don't seem to work.

> I don't have an example for R^2, but if n > 2 you can do this:
> Take any g in C^oo with support near 0 and g(0) nonzero. For b > 0, set g_b(x) = g(bx). Then int |g_b|^2 = (int |g|^2)/b^n and int |D(g_b)|^2 = (int |D(g)|^2)/b^(n−2). (All integrals are over R^n and D is any partial derivative of f.) Letting u denote any unit vector you like, define f(x) = sum (j=1,oo) g_(2^j)(x − ju). Then f is C^oo, and f and any Df belong to L^2. But f(ju) = g(0) for all j, so f does not tend to 0 at infinity.

Huh. WIthout thinking too hard about it, I assumed that this construction gave a counterexample for R^2. I guess it doesn't.

You can do something similar in R^2. First a one−variable fact:

Suppose that a > 0 and psi(t) = phi(t^a) for t > 0. Then a change of variables shows that

int_0^infinity t |psi'(t)|^2 dt

= a int_0^infinity t |phi'(t)|^2 dt

(you may want to check that...).

For x in R^2 and a > 0 let x^a be the vector pointing in the same direction with |x^a|
= |x|^a:

x^a = |x|^(a−1) x.

Now suppose that g is a smooth function in R^2, g(0) <> 0, and g has support in the unit ball. Also assume that g is constant in some neighborhood of the origin. And assume that g is radial. For a > 0 let

g_a(x) = g(x^a).

Then ||g_a||_2 −> 0 as a −> infinity, and integrating in polar coordinates and using the one−variable fact shows that the L^2 norm of the gradient of g_a also tends to 0 as a −> 0.

So a sum of translates of suitably chosen g_a's gives a counterexample.

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