

Re: f and f' square integrable

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- *From:* David C. Ullrich <ullrich@xxxxxxxxxxxxxxxxxxxx>
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On Sat, 06 Jan 2007 18:50:23 GMT, Stephen Montgomery-Smith
<stephen@xxxxxxxxxxxxxxxxxxxx> wrote:

David C. Ullrich wrote:

On Sun, 31 Dec 2006 14:32:45 -0800, The World Wide Wade
<waderameyxiii@xxxxxxxxxxxxxxxxxxxxxxxx> wrote:

In article <NDHlh.1114208\$084.1066831@attbi_s22>,
Stephen Montgomery-Smith
<stephen@xxxxxxxxxxxxxxxxxxxx> wrote:

Fedor wrote:

Hi all,

suppose that $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is
a smooth function such that
 f^2 and (f')
 $)^2$ are integrable over \mathbb{R}^n .
Is it true that $f(x)$ tends
towards 0 when
 $|x|$ tends towards infinity ? It
is easy for $n=1$ but is it true
for the
general case ?

Regards,
fedor

I think not.

Re: f and f' square integrable

First consider a function like $f(x) = \log(1/|x|)^a$ for $0 < a < 1/2$. Check that f and f' are square integrable in \mathbb{R}^2 .

Next consider $\sum a_n f(x - (2n, 0))$ where a_n is square summable.

If $f(x) = [\log(1/|x|)]^a$ for x near 0, then f is not smooth at 0. Perhaps you can play around with this to arrive at an example, although the obvious things don't seem to work.

I don't have an example for \mathbb{R}^2 , but if $n > 2$ you can do this: Take any g in C^∞ with support near 0 and $g(0)$ nonzero. For $b > 0$, set $g_b(x) = g(bx)$. Then $\int |g_b|^2 = (\int |g|^2)/b^n$ and $\int |D(g_b)|^2 = (\int |D(g)|^2)/b^{n-2}$. (All integrals are over \mathbb{R}^n and D is any partial derivative of f .) Letting u denote any unit vector you like, define $f(x) = \sum_{j=1, \infty} g_{(2^j)}(x - ju)$. Then f is C^∞ , and f and any Df belong to L^2 . But $f(ju) = g(0)$ for all j , so f does not tend to 0 at infinity.

Huh. Without thinking too hard about it, I assumed that this construction gave a counterexample for \mathbb{R}^2 . I guess it doesn't.

You can do something similar in \mathbb{R}^2 . First a one-variable fact:

Suppose that $a > 0$ and $\psi(t) = \phi(t^a)$ for $t > 0$. Then a change of variables shows that

$$\begin{aligned} \int_0^\infty t |\psi'(t)|^2 dt \\ = a \int_0^\infty t |\phi'(t)|^2 dt \end{aligned}$$

(you may want to check that...).

For x in \mathbb{R}^2 and $a > 0$ let x^a be the vector pointing in the same direction with $|x^a| = |x|^a$:

$$x^a = |x|^{a-1} x.$$

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Now suppose that g is a smooth function in \mathbb{R}^2 , $g(0) \neq 0$, and g has support in the unit ball. Also assume that g is constant in some neighborhood of the origin. And assume that g is radial. For $a > 0$ let

$$g_a(x) = g(x^a).$$

Then $\|g_a\|_2 \rightarrow 0$ as $a \rightarrow \infty$, and integrating in polar coordinates and using the one-variable fact shows that the L^2 norm of the gradient of g_a also tends to 0 as $a \rightarrow \infty$.

So a sum of translates of suitably chosen g_a 's gives a counterexample.

David C. Ullrich

Hi David,

I just looked over your example. It really is rather beautiful.

Just noticed a typo – of course the first "as $a \rightarrow \infty$ " should be "as $a \rightarrow 0$ ".

I like it because it's "soft" – doesn't rely on the specific norm of a specific function, works just because something tends to zero, the rate at which it tends to zero being irrelevant.

Also it's seemed to me for some years that a sum of translates of dilates of one function "should" give an example here if one existed – the last time the question came up I noticed myself that that didn't work and ended up giving an example based on the logarithm. Nice to see that the idea does work, with a sort of quasi-dilation in place of an ordinary dilation. Only remaining problem is that I really don't feel I understand why it works – that one-variable fact is just a meaningless calculation, as far as my current understanding goes.

Stephen

Re: f and f' square integrable

David C. Ullrich

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