

Re: Did you hear about Euler–Mascheroni integrals?

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Joachim Selke <[selke@xxxxxx](mailto:selke@xxxxxx)> wrote:

Hi!

While doing some computation tasks I recently came across an interesting family of integrals, defined by (for natural numbers  $n$ )

$$I(n) = (-1)^n * \int_0^{\infty} [\ln(z)]^n * \exp(-z) dz.$$

The  $(-1)^n$  factor is there because the integral is positive for even  $n$  and negative otherwise.

Let's look at some values of  $I(n)$ :  $I(0)$  is 1,  $I(1)$  is the Euler–Mascheroni constant  $\gamma$ . Additionally,  $I(n)$  seems to converge to  $n!$ , for large  $n$ . To be more precise, by experiments I guess that  $I(n) / n!$  tends to 1.

Your guess is correct. The most important part of the integral lies near 0, and so, for large  $n$ , a reasonable approximation of  $I(n)$  can be obtained by replacing  $\infty$  by 1 and  $\exp(-z)$  by 1, giving

$$(-1)^n * \int_0^1 [\ln(z)]^n dz$$

which is indeed  $n!$  .

For a better approximation, use more terms of the Maclaurin expansion of the exponential. E.g., using just two terms, we get the improved approximation

$$n! (1 - 1/2^{(n+1)})$$

As a numerical check,  $I(16) = 2.092263034... * 10^{13}$   
while  $16! (1 - 1/2^{17}) = 2.092263026... * 10^{13}$   
and  $16! = 20922789888000$

David W. Cantrell

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Also note that  $I(n)$  is equal to the absolute value of  $\Gamma^{(n)}(1)$ , where  $\Gamma^{(n)}$  is the  $n$ -th derivative of the Gamma function.

By chance, I found a Mathworld entry that deals with  $I(n)$  and calls it Euler–Mascheroni integrals:

<<http://mathworld.wolfram.com/Euler–MascheroniIntegrals.html>>.

Unfortunately, this entry does not mention who coined this name for  $I(n)$  and gives no further references. I already asked the MathWorld team a few weeks ago but received no answer yet.

I'm interested in more details about this family of integrals but was not able to find anything, except for some web pages I describe below. I even looked in some mathematical encyclopedias but still did not find anything.

This is where you come into play: Maybe you have heard the name "Euler–Mascheroni integrals" before. If so, then you can make me happy by sending me references to books or scientific articles that either mention this term or deal with some properties of  $I(n)$ , for  $n \geq 2$ . Please let me know. :-)

Thanks,  
Joachim

Now the web pages I found by myself:

<<http://numbers.computation.free.fr/Constants/Miscellaneous/gammaFunction.html>> The term is used there but again without reference. I tried to contact the authors of this page but got no response.

<<http://mathworld.wolfram.com/ExtremeValueDistribution.html>>  
Again MathWorld.  $I(n)$  seems to be connected to the raw moments of the Gumbel distribution. But looking for papers about this distribution also did not help me.

<[http://www.math.ku.dk/~richard/courses/binf\\_project/statistics.pdf](http://www.math.ku.dk/~richard/courses/binf_project/statistics.pdf)>  
The term also is used here. I contacted the author and he told me that the only reference he knows about is the MathWorld entry.