Re: Accumulation points in metric space

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- *From*: "Dave L. Renfro" <renfr1dl@xxxxxxxxx>
- *Date*: 21 Feb 2007 15:14:15 −0800

Dave L. Renfro wrote (in part):

> If (X,d) is not separable, then there exists an uncountable subset having no accumulation point in X. In fact, I believe even more is true: If (X,d) is not separable, then there exists an uncountable subset that is uniformly isolated (i.e. there exists epsilon > 0 such that the distance between each pair of distinct points in the uncountable subset is at least epsilon).

I happen to be at home now, where my notes are. Here's the theorem and proof.

THEOREM: If (X,d) is a non−separable metric space, then there exists epsilon > 0 and an uncountable subset E of X such that, for each pair of distinct points x & y in E, we have d(x,y) > epsilon.

PROOF: I first claim there exists a positive integer n such that for all countable subsets B of X we have (for all x in X)(for all b in B)(d(x,b) > epsilon). If not, then for all positive integers k there exists a countable subset B_k of X such that the distance between each pair of points, one point chosen from X and one point chosen from B_k, is less than or equal to epsilon. This contradicts the non−separability of X, since UNION(k=1 to oo) B_k is a countable dense subset of X. I will now construct an appropriate set E, of cardinality aleph_1, by transfinite induction over the ordinals less than omega_1.

Choose any x in X and denote this point by x_1. Let alpha > 0 be a countable ordinal and assume that, for each ordinal beta < alpha, we have chosen a point x_beta such that the distance between each pair of distinct points in the set {x_beta: beta < alpha} is greater than 1/n. Since {x_beta: beta < alpha}

is a countable subset of X, and there exist points
in X not belonging to this countable set (it trivially
follows from the non−separability of X that X is
uncountable), it follows from what I proved earlier
that there exists x_alpha in X such that each of the
distances between x_alpha and the points belonging
to {x_beta: beta < alpha} is greater than 1/n.
The set E = {x_beta: alpha < omega_1} is a set of
cardinality aleph_1 such that any two of its distinct
points are a distanc