Re: Questions about Algebraic Functions

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- *From*: Robert Israel <israel@xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx>
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"John L. Barber" <jlbarber@xxxxxxxx> writes:

> I have several questions regarding algebraic functions that I'm rather desperate to find the answers to for a paper I'm working on. A search of the literature and the internet doesn't yield anything relevant, so I thought I'd ask here.
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> I'd previously asked some of these questions in this old thread:
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> http://mathforum.org/kb/message.jspa?messageID=5659683&tstart=0
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> My apologies for starting a new thread, but there seemed to be little interest in the old one.
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> Consider a function f of a (possibly complex) variable z. (For now, let's just consider functions of a single variable.) It is my understanding that f(z) is said to be an algebraic function if there exists a function F(z, w) that is a polynomial in z with coefficients which are polynomials in w, such that F(z, f(z)) = 0. Furthermore, it is my understanding that, given F(z, w), one can find the roots of the original function f(z) by searching for the roots of F(z, 0) = 0.
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> My first set of questions concerns the nature of the function F(z, w), and my second set concerns the nature of the roots.
>
> * The function F(z, w) *
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> It is clear that, given an algebraic f(z), the function F(z, w) is not uniquely defined. Multiplying any applicable F(z, w) by any (well, almost any) other function G(z, w) will yield another function H(z, w) = G(z, w)*F(z, w) that also satisfies H(z, f(z)) = 0. Furthermore, there may be roots of H(z, 0) which are not roots of f(z). So my question is:
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> (1) Does there exist an "optimal" F(z, w) for every algebraic function f(z), such that all of the roots of F(z, 0) are roots of f(z) and vice versa? If so, how do we find and/or identify it?

It sounds like you're talking about a single−valued function, not the "complete algebraic function" which is multivalued. The other roots of F(z,0) are roots of the to the other branches of the complete algebraic function.

> (2) As an example, consider the function f(z) = sqrt(1+z) + sqrt(1+2z). As I understand it, this is algebraic. However, straightforward algebraic machinations yield F(z, w) = (w^2 − 2 − 3z)^2 − 4(1+z)(1+2z) as an associated F such that F(z, f(z)) = 0. Yet, clearly this F is wrong, since the roots of F(z, 0) = z^2 = 0 are two 0's. 0 is not a root of the original f(z). What have I done wrong, and what is the right F for this f?

You've done nothing wrong, you've merely shown that there are no roots of f(z). 0 is a root of other branches of f(z).

> * The nature of the roots *
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> Consider now an algebraic function f such that all of the coefficients contained therein are real. Let's call the vector of parameters (i.e. coefficients) upon which f depends "p". p is a finite list of real numbers. Let's explicitly show the dependence of f upon these parameters: f = f(z, p). For any given value of p, f will have some set of n (possibly complex) roots r(p) = {r1(p), r2(p), ..., rn(p)}. In general, there may be some multiplicity of roots, so that elements of r(p) may be repeated. My questions:
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> (1) Must n be finite for every p?

Not for those p that make f(z,p) = 0. Otherwise yes.

> (2) Must r(p) be a continuous function of p? In other words, can one of the "root branches" ri(p) jump dicontinuously to another value at some particular value of p?

You can have $r_i(p)$ −> infinity at some value of p. For a simple example, f(z,p) = 1 − p z so $r_1 = 1/p$.

> (3) Must n be the same for every p? In other words, must f(z, p) have the same number of roots, regardless of the values of the coefficients upon which it depends?

No. Same example.

> It is this last question that concerns me. I have a rather complicated algebraic function which I've been studying the roots of numerically. I've identified 2 "root branches". My problem is that one of these branches seems to stop abruptly at a particular value of the parameter*, so that for a particular regime of parameters there is only one root branch. I'm wondering if this is "allowed" or if I'm just missing a root somehow.

I suspect that the missing root belongs to the "wrong" branch of your algebraic function.

Consider f(z,p) = (sqrt(z)−1)^2 − p. Thus for a root we need sqrt(z)−1 = (+/−) sqrt(p), and thus z = (1 (+/−) sqrt(p))^2. But, depending on which branch you're using, "the" square root of the right side might be 1 (+/−) sqrt(p) or −1 (+/−) sqrt(p). If you use the principal branch of the square root, (1−sqrt(p))^2 is a root of f when 0 <= p <= 1, but not if p > 1.

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