Re: Problem on an nxn grid

Source: http://sci.tech−archive.net/Archive/sci.math/2007−05/msg02035.html

- *From*: Chip Eastham <hardmath@xxxxxxxxx>
- *Date*: 11 May 2007 11:13:58 −0700

On May 11, 1:23 pm, Jonathan Berry <jbe...@xxxxxxxxxxxxx> wrote:

> On May 11, 8:47 am, Chip Eastham <hardm...@xxxxxxxxx> wrote:
>
> > On May 10, 11:22 pm, Jonathan Berry <jbe...@xxxxxxxxxxxxx> wrote:
> >
> > > An nxn grid is filled with positive integers. I want a practical
> > > way
> > > (for large n, say up to 300) to select n/2 integers from the
> > > grid such
> > > that the sum of the integers is the smallest.
> > >
> > > The limiting factor on the selection is that each coordinate
> > > from 1 to
> > > n is used exactly once. So, for example, if the grid entry at
> > > (3,
> > > 149) is one of the integers, no other selection may be of the
> > > form (3,
> > > y) nor (x, 149), nor (x, 3), nor (149, y).
> > >
> > > I'll have a computer to do the crunching, but when I
> > > considered the
> > > "brute force" method of just trying every combination, it
> > > looked like
> > > n! / (n/2)! operations, so the computer and I could be waiting
> > > a long
> > > time. 42, and all that.

For example? A 6x6 grid, the smallest sum of
3 elements

1 5 8 2 1 3
2 9 7 3 4 5
1 6 7 2 4 2
3 2 5 1 3 1
8 6 2 1 4 2
3 2 7 4 1 1

Numbering from the left and top, a solution is
(5, 1) + (2, 6) + (4, 3) =
1+2+1 = 4.
No numbers on the diagonal (such as (1,1), (4,4), or (6,6)) can be
selected, because that would contradict the condition that each
coordinate can be used only once. So the value of any (a,a) number on
the grid is irrelevant.

Believe it or not, this is a model of a practical problem!

TIA.

--
Jonathan Berry

It's an interesting problem, so I admit it's a petty observation to point out that your solution of the example problem is incorrect. It is customary to provide the row coordinate first and column coordinate second, but for consistency I'll try to use your notation.

You say the grid entries are numbered from the top and left, so presumably your (5,1) entry is claimed to be 1 based on the 5th column, 1st row being 1. Likewise the claim of (2,6) entry being 2 appears to correspond to 2nd column and 6th row. But neither (4,3) entry interpreted as 4th column, 3rd row nor vice versa contains an additional 1 (though a 2 appears in the 4th column and 3rd row).

Perhaps you had these entries in mind which give a minimal sum of 4:

1st col, 3rd row = 1
2nd col, 4th row = 2
5th col, 6th row = 1

The minimal solution is not unique, for example:

1st col, 3rd row = 1
2nd col, 6th row = 2
4th col, 5th row = 1

A slightly more useful observation: Without loss of generality, you may symmetrize the grid by replacing A(i,j) by the least of A(i,j) and A(j,i) for all distinct i,j. This follows from the observation that in any feasible choice of n/2 entries, replacing A(i,j) by A(j,i) is again feasible (no duplicated coordinates).

It appears that branch and bound methods for these problems will be effective. If you have a "real world" size problem you'd like me to tackle, email me and I'll try a quick Prolog program on it.

regards, chip

> Thank you, Chip, for so gently pointing out the drawbacks in
> my analysis of the example. I saw 2 at (4,3), but typed
> 1. Then I overlooked that there were two "other" solutions.
> Kind of like a chess player giving away the queen, then both
> rooks, in the same game. Or to warp another saw, "Make
> up an example on the fly, then ... live with it".
>
> I am not sure that your observation about symmetrizing
> the grid is helpful to me. In the application that I have in mind
> (which is the pairing of chess tournaments!), the important
> part is the coordinates (and ij is importantly different from ji), not
> in the value of the cells nor in the value of the sum, except that it
> be minimized.
>
> It is slightly concerning that there may be more than one solution,
> but if larger numbers are used, then that becomes less likely.
>
> Thank you, but I'll hold off on your offer of writing a Prolog program
> for now. I'm trying to get everything under one "roof",
> in PowerBASIC. But I may be out of my depth here. So
> "for now" may be very short.
>
> I only partly understood the algorithm described by Ali (Asinop). I'm
> OK until:
>
> "− Then form a graph with n nodes representing coordinates. Place
> edges between nodes i and j (i<j) with weight a'_{ij}.
> − Do maximum weighted matching on this graph."
>
> I'd need to learn what graph, node, weight, and (maximum) weighted
> matching are. And then translate that into plodding computer code.
>
> Ironically, I do hold a Math bachelor's from the same institution
> where Robert Israel teaches. You might call it a "No−More Analysis−My−
> Head−Hurts" degree. Likely to his credit, my degree was granted while
> he was still at Princeton.
>
> −−
> Jonathan Berry

A reference that I have to hand for this sort of stuff is:

Algorithmic Graph Theory, by Alan Gibbons
Cambridge University Press, 1985

a slender paperback volume that addresses both
theory and algorithmic issues. Chapter 5 is on
matchings (selecting edges of a simple graph so
that no node appears more than once), and Sec.

5.3 is on maximum−weight matchings and an algorithm for these problems proposed by Edmonds and Johnson:

Edmonds, J. and Johnson, E.
"Matching: A well−solved class of integer linear programs", Combinatorial Structures and Their Applications, Gordon & Breach, NY
pp. 89−92, 1970

Your application may still depend on distinguishing A(i,j) and A(j,i), since you can always discern the least of these after the fact of solving the reduced problem, as proposed by asinop as well, that considers only a minimum sum using the lesser of each pair. You're only setting the information aside, not ignoring any distinction important in your application.

regards, chip

.