Re: Dedekind Cuts, Fundamental Sequences: why?

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- *From*: James Burns <burns.87@xxxxxxx>
- *Date*: Mon, 04 Jun 2007 18:21:04 −0400

Hatto von Aquitanien wrote:

> David C. Ullrich wrote:
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> > On Mon, 04 Jun 2007 05:37:27 −0400, Hatto von Aquitanien
> > <abbot@xxxxxxxxxxxxxx> wrote:
> >
> > > David C. Ullrich wrote:
> > >
> > > > The fact that the reals are complete is used in
> > > > many
> > > > places − analysis simply would not work
> > > > without it. You're
> > > > just at the start...
> > >
> > > That begs the question.
> >
> > Yes it does. When you know some analysis it will be very clear
> > to you why completeness is so important.
>
> That still begs the question. The fact that something _is_
> useful does not explain _why_ it is useful.

Please excuse me for butting in here, Dr. Ullrich.

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Whether it begs the question depends on which question
is on the table. Judging from the subject header, your first
question seems to have been "Why are Dedekind cuts used to define
reals?" (And partitions of Cauchy sequences as well?)

That question seems to have been answered well, by several people, not just Dr Ullrich.

However, it happens that I, too, have wondered about the other question, the question behind your question, or, at least, the question I think you're asking. It may be that my thoughts on the topic will be useful to you, even if only as an example of what you are not asking about. It may also be that someone reading sci.math will clarify or correct my thoughts, something I would appreciate.

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Personally, I find the representation of the reals as a partition of Cauchy sequences more illuminating than as the set of Dedekind cuts. (Disclaimer: I'm not aware of anything historical that would suggest my opinions on the connection between algorithms and the "origins" of real numbers is anything more than opinion.)

Imagine some broad category of numerical algorithms, each intended to calculate a single number (leaving aside what "number" means,for the moment). If we abstract away all the details of one of the algorithms, what we are left with is one rational number followed by another (rational because each step of the algorithm takes finite time, so we must finish writing down our representation of each number), followed by another, and so on without end. Thus, we can represent each algorithm (with specific initial conditions) as a sequence of rational numbers.

The best kind of algorithm would be one which produces intermediate results with continue to agree with each other to any specified level of accuracy, once the algorithm has operated long enough. (Complete agreement after some point would imply that the particular value was better than every other possible result. This may not be possible in some cases, such as calculating sqrt(2), and so should not be required.) Thus we can represent these "Best of Breed" algorithms as Cauchy sequences of rationals.

Consider the case of two different algorithms that report intermediate results completely disagreeing with each other at every point, and yet, given any specific precision to which they should agree, they will agree and continue to agree (within that precision), once they have both been operated long enough. Can we say that they calculate "the same number" if the actual numbers they calculate disagree completely?

Because it is too, too convenient to give up saying
these two algorithms calculate the same number and
imagining that number as a dimensionless, structureless
point on the real number line, we do say that they
calculate "the same number". What is this number they
calculate? In my view, that dimensionless, structureless
point is "really" the set of all the good algorithms
that calculate its position, which is to say, the equivalence
class of Cauchy sequences that converge to that same point.

And what is that point again? The Cauchy sequences
that converge to it. It all seems very circular, but
built up, step by step, from rationals, to sequences
of rationals, to Cauchy sequences, to partitions of
Cauchy sequences, it's all well−founded and consistent.

Jim Burns
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