Re: Dedekind Cuts, Fundamental Sequences: why?

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In article <FqWdncA−GfEgP_HbnZ2dnUVZ_g−dnZ2d@xxxxxxxxxxxxx>,
Hatto von Aquitanien <abbot@xxxxxxxxxxxxxx> wrote:

> Dave Seaman wrote:
>
>> On Sat, 09 Jun 2007 01:34:05 −0400, Hatto von Aquitanien wrote:
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>>> "Cauchy's Criterion for Convergence. In accordance with Cantor's basic
>>> idea, real numbers can be described by convergent rational sequences.
>>> Two rational sequences (r_n) and (s_n) have the same (real) limit, if and
>>> only
>>> if the sequence of their differences (r_n−s_n) converges to zero. It is
>>> natural therefore to _define_ the _real_ _numbers_ as equivalence classes
>>> of _convergent_ _rational_ _sequences_ ; two sequences being equivalent
>>> when their difference sequence converges to zero. _For_ _this_
>>> _definition_ to be meaningful, the _convergence_ of a sequence has to be
>>> _characterized_ _without_ making _use_ of _limits_ . This can be done
>>> with the help of Cauchy's criterion, which will be used to define the
>>> sequences concerned."

> http://www.amazon.com/Numbers−Graduate−Texts−Mathematics−Readings/dp/0387974970
>
>> Although the consensus in sci.math seems to be otherwise, the way I actually learned it in my graduate real analysis course is that we con't speak of "Cauchy sequences of rationals" when constructing the real numbers. We speak of "fundamental sequences of rationals" instead. I believe this is Cantor's terminology.
>>
>> A sequence {a_n} of rationals is said to be "fundamental" if, for every *rational* epsilon > 0, there exists N > 0 such that | a_m − a_n | < epsilon for every m,n > N.
>>
>> The reason for this is that the definition of a Cauchy sequence makes sense only in a metric space, and the definition of a metric space requires the metric to be a real−valued function.
>
> The proof for convergence doesn't seem to require that distance be more than rational if the limit is rational. Let {a_n} be a fundamental sequence of rational numbers, i \in \N such that for all m,n>i => abs(a_m−a_n)<epsilon_i, where epsilon_i \in \Q. If for some q \in \Q and for all Epsilon \in \Q and Epsilon > 0 we can find some j \in \N such that abs(a_n−q)<Epsilon for all n>j, then we can say that {a_n} converges to the limit q. We don't need real numbers for that, do we?
>
>> Therefore, there are
>> no metric spaces and no Cauchy sequences until *after* we have finished constructing the reals, if we are to avoid circularity.
>
> Stoll[*] introduces Cauchy sequences of rational numbers without first defining the real numbers. I haven't read the entire development, but it doesn't appear that he talks about limits until he defines the real numbers. I don't believe he ever uses the term "convergence".
>
> If I read III §1.2.2 (pp 2−4) correctly, the field of rational numbers satisfy the conditions of a Hausdorff space, and therefore allows a series to converge to a unique limit.

But does not require it to do so. Even a Cauchy seqeunce does not have to have a limit in a arbitrary Hausdorff space.

> As I mentioned to Virgil, Pickert and Görke attempted to make their development self−contained with as little reliance on results from external theories as possible. That is my reason for not wanting to admit metric

> spaces into the development. I don't believe it is logically necessary. It may be a satisfying and insightful way of proceeding, but I don't believe it is the only way.

Without a topology, one cannot even define either a Cauchy sequence or convergence. And the standard topologies of number systems all seem to devolve from metrics.

> I may have been mistaken in what I said to Virgil regarding the point at which they actually used the Cauchy criterion in connection with convergence. That is, in proving the properties 61_1, 61_2 and 61_3 (page 141) which seem to follow from the definition of real number addition and multiplication ((59) on page 139).
>
> http://baldur.globalsymmetry.com/open−source/org/sth/math/behnke−et−al/vol−1.djvu
>
> I wish they had simply provided the proof, or at least given an explicit reference to where the proof, in the form they intend, can be found. I'm pretty sure I can prove it using the Cauchy criterion without requiring any assumption not yet stated. It seems far simpler to use (62) on page 141 with u=0. Am I missing something?

Without a topology, one cannot even define either a Cauchy sequence or convergence. And the standard topologies of number systems all seem to devolve from metrics.

> Every time we turn on our computers, they do something called booting. Booting is shortened from "bootstrapping" which is a euphemism meaning to lift oneself by one's own bootstraps. A similar situation arrises when we get the latest source code for gcc (the gnu compiler collection). In order to get a compiler from the source code we have to compile the compiler. The only way to compile the compiler is with a compiler. These are all examples of a self−referential system. The ultimate example of the self−referential problem in Russell's Paradox. I am quite confident that such a self−reference is at the core of the difficulty in defining the real numbers in terms of the rational numbers.

If one needs to bootstrap one's mathematics, it must be something like setting up the axioms for a set theory, which would occur before any consideration of numbers has even begun.
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