Re: A question in noetherian ring!

Source: http://sci.tech−archive.net/Archive/sci.math/2007−07/msg05230.html

- *From*: galathaea <galathaea@xxxxxxxxxx>
- *Date*: Mon, 30 Jul 2007 15:56:51 −0700

In article <kk0ra3t5cc10627g57kkb6b2817gtut5j0@xxxxxxx>, quasi <quasi@xxxxxxxx> wrote:

!! On Mon, 30 Jul 2007 00:32:28 −0400, quasi <quasi@xxxxxxxx> wrote:
!!
!! >On Sun, 29 Jul 2007 21:16:50 −0700, Cooper <cooper0040@xxxxxxxxx>
!! >wrote:
!! >
!! >>On 7 30 , 11 14 , quasi <qu...@xxxxxxxx> wrote:
!! >>> On Sun, 29 Jul 2007 09:33:04 −0700, Cooper <cooper0...@xxxxxxxxx>
!! >>> wrote:
!! >>>
!! >>> >Hi!
!! >>>
!! >>> >The question I want to ask is following;
!! >>>
!! >>> >Let M be a finitely generated R−module, where R is noetherian. Suppose
!! >>> >I is an ideal of R such that for each element a in I, there exists a
!! >>> >nonzero element x in M s.t. xa=0. Show that xI=0 for some nonzero
!! >>> >element x in M.
!! >>>
!! >>> Initially, I was almost sure the above claim was false, but after
!! >>> several failed attempts to produce a counterexample, I'm no longer so
!! >>> certain. Perhaps it's true, but I remain skeptical.
!! >>>
!! >>> Was this an exercise from a text? If so, which?
!! >>>
!! >>> quasi
!! >>
!! >>Yes. It is an exercise problem contained in chap 27 of Issac Martin's
!! >>Algebra text(of course graduate course)
!! >
!! >Ok, in that case, I'll stop trying to disprove it.
!! >
!! >Can I assume that R is a commutative ring with 1?
!! >
!! >But R is not necessarily an integral domain, right?
!! >
!! >Also, as a minor correction, xI should probably be written as Ix or
!! >I*x (the ring elements multiply on the left).

!!
!! Ok, 2 references ...
!!
!! Reference (1):
!!
!! Linear Systems over Commutative Rings
!! Brewer / Bunce
!! Marcel Dekker, 1986
!!
!! Theorem 1.7 (pages 20−21)
!!
!! Reference (2):
!!
!! Introduction to Commutative Algebra
!! Atiyah / MacDonald
!! Addison−Wesley, 1969
!!
!! Proposition 1.11 (page 8)
!!
!! Here are the statements ...
!!
!! Theorem 1.7 (Brewer / Bunce):
!!
!! Let R be a noetherian ring with M a finitely generated R−module. Then
!! the set of elements of R which are zero divisors on M is a finite
!! union of prime ideals of R. Moreover, if P_1, ..., P_n is the set of
!! prime ideals involved, then each P_i is the annihilator of a single
!! nonzero element of M.
!!
!! Proposition 1.11 (Atiyah / MacDonald):
!!
!! Let I be an ideal contained in the union of the prime ideals P_1, ...,
!! P_n. Then I is contained in one of the ideals P_i.
!!
!! The proofs can be obtained using Google book search.
!!
!! Once you have the theorems above, the result you want is immediate.

i hope you don't mind
but i'd like to develop this for my own understanding

it appears that 1.7 is the main step being stated for the proof
(1.11 is a more standard result on prime ideals in general)

the proof using these is:
we are given noetherian ring with finitely generated module
by 1.7 all zero divisors on M are members of a union P_1 U P_2 U ... U P_n
we are given an I where for each a_i e I thereExists x_i e M with x_i a_i = 0
therefore I is in P_1 U P_2 U ... U P_n and by 1.11 I c P_i for some i
by 1.7 again, each P_i is the annihilator of a single element x of M
and qed xI c xP_i = 0 for this element of M

now how is theorem 1.7 proven?

well every annihilator a in R of an element x of M
is contained in a maximal annihilator A in R for this element of M because R is noetherian

A is prime because
take any product of elements from R, (r_1 r_2) which lies in A
if r_1 is not in A, then x r_1 =/= 0 and we can call the annihilator ideal of this element A_2
but A_2 contains A since A annihilates on m and by maximality A_2 = A
and since (x r_1) r_2 = x (r_1 r_2) = 0
r_2 is an annihilator in A
and the condition if (r_1 r_2) in A −> r_1 in A or r_2 in A
is equivalent to A prime

the number of such elements x for which an A(x) annihilates is finite
because the module is finitely generated
and any linear relation among the module elements annihilated
(which would be immediate if they were to exceed the number of bases elements)
would imply that the primes sets were not maximal over the elements they annihilate
(a linear relationship translates to membership in the disjunction of the ideals)

okay
this seems pretty clear to me
(still, please let me know if i have understood anything incorrectly)

it seems like overkill to me
but i now don't see any alternative
and i do like the elegance of the appearance of primes to prove uniqueness

still...

annihilators are always ideals
this was the primary point that led me to my first ideas in this thread

ideals in noetherian rings have a finite collection of generators

my original suggestion was to follow the elements x_i of M that annihilate the generators
through to the induced quotient of M by I
and notice that this maps all x_i to a single element
but this maps the annihilators trivially to 0 and does not give the information i had hoped
(this now seems too stupidly obvious to repeat in public − but there you go)

however
it seems like you should be able to prove any two elements of I annihilate the same x
without proving that the annihilators are prime
but i do not see it right now...

anyway
i was planning on studying the nullstellensatz again in august
(this time categorially and focussing on the many relations between algebraic sets and ideals)

and this is a great problem to kick off with which already seems intimately related

thanks both the op and quasi for this great problem and the other ideas it brings!

−=−
galathaea: prankster, fablist, magician, liar
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