Rational numbers, irrational numbers: each dense in real numbers

Source: http://sci.tech−archive.net/Archive/sci.math/2007−09/msg04029.html

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- *Date*: Tue, 18 Sep 2007 12:21:21 −0700

In ZFC, with standard definitions of the real, rational, and irrational numbers, let p_i be an irrational number between zero and one for i from a suitably large well−ordered index set X. With the well−ordering of the index set, let the i'th element p_{i+1} be an irrational number between zero and p_i, where i+1 is the least element of the well−ordering X_i setminus i, that is defined to equal X_{i +1}. There are uncountably many irrational numbers less than each p_i and greater than zero, else the irrational numbers are countable (or the real numbers are not standard). Define P to be comprised of the p_i's. There exists a rational number q_i between p_i and p_{i+1}, else the rational numbers are not dense in the reals thus that between any two irrational numbers there is a rational number. For each of the irrational p_i's, there thus exists at least one unique rational q_i between p_i and p_{i+1}, and infinitely many. Let the ordered pair (p_i, q_i) be an element of a function, as a set, from P to Q. If there is an uncountable set P of irrational numbers in (0,1), then there is a 1−to−1 function defined by the set {(p_i, q_i), i E X} from uncountable P to a subset of Q the rational numbers, and there is thus an injection from an uncountable set of irrational numbers to a subset of the rational numbers, a subset of a countable set is countable.

Contradiction ensues, from that an uncountable set injects into a countable set. Which presupposition is false? Perhaps it is so that the irrationals can not be well−ordered in their normal ordering, but, via separation, for any given subset of the irrational numbers in (0, p_i) there exists (0, p_{i+1}) for any p_i such that 0 < p_{i+1} < p_i, or the irrationals are not dense in the reals. Perhaps it is so that there are no uncountable subsets of the irrationals in (0,1), but then the irrationals wouldn't be uncountable. The rationals are dense in the reals so between any distinct p_i and p_{i+1} there exists a q_i, or p_i = p_{i+1} and p_i =/= p_{i+1}. In ZFC there exists a suitably large well−ordered index set.

Quantify over sets, ZF(C) and/or the standard definitions of the real, rational, and irrational numbers are thus inconsistent.

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