Re: Rational numbers, irrational numbers: each dense in real numbers

Source: http://sci.tech−archive.net/Archive/sci.math/2007−10/msg02802.html

---

- *From*: "Ross A. Finlayson" <raf@xxxxxxxxxxxxxxx>
- *Date*: Sat, 13 Oct 2007 15:54:28 −0700

On Oct 12, 11:38 am, rem6...@xxxxxxxxx (Robert Maas, see http://tinyurl.com/uh3t) wrote:

> > From: "Ross A. Finlayson" <r...@xxxxxxxxxxxxxxx>
> > Among the notions of why there are "more" irrationals than rationals is the (not fundamentally) heuristic notion that if one were to try to sample at uniform random from the real numbers in [0,1] by flipping fair coins (independent Bernoulli trials) to form the binary expansion of a real number, that it is extremely unlikely to have the sequence terminate in ending with all zeros or ones, or some repeating sequence.
>
> What you said there is empirically meaningless. There is no way even in principle to determine whether an infinite sequence of coin flips eventually enters a repeating loop. Even if you use a quantum−mechanical process to emulate coin flips, repeating the starting conditions for a quantum flip *exactly*, such as inside a Shroedinger−cat box totally isolated from outside influences, cooled to Bose condensate to eliminate noise, still the theory of quantum mechanics AFAIK is consistent with that experiment eventually going into a loop. For example, the whole Universe could degenerate into a thermal equilibrium (outside that somehow magically protected Shroedinger−cat box) and then some earlier state of the Universe could repeat, and QM could be a function of the whole−Universe state, whereupon the whole Universe, including the innerds of the Shroedinger−cat box, begin repeating exactly the same activities that they previously did from the same initial state before.
>
> From an empirical viewpoint, the best you can learn is an initial segment of the sequence, with no knowledge of what will follow later.
>
> I personally consider it unlikely that a sequence of identical QM flips will produce a rational number, but I'm not a god who can *know* such a thing. I personally consider it reasonable that a sequence of identical QM flips would have some *computable* number

pre−programmed into it, hence the totality of such possible sequences would be countable
− in the weak sense that it's a Turing−undecideable subset of a countable set (the set of all syntactically−correct coin−flip algorithms can be enumerated, but only some of those algorithms actually generate an infinite sequence of flips, the rest either halt or run forever after generating their very last coin−flip event, and it's undecideable whether a given algorithm will indeed generate an infinite sequence of flips or not),
− or in the other weak sense that for any machine that will at some point produce its very last coin−flip we simply define it to produce an infinite sequence of all−heads flips after that point, in which case we have no decision procedure for knowing where we need to do this trick and where we should just be more patient.
And in either case, there is no decision procedure for whether two different algorithms produce forever−identical or eventually−different output, hence no way to eliminate duplicate sequences in order to achieve a true enumeration of all computable sequences. So in fact there is no Cantor−rigor method to enumerate the computable reals. (There *is* a Cantor−rigor method to enumerate the rationals, a zigzag path through Z x Z+ where any fraction with nontrivial GCD of numerator and denominator is eliminated to get rid of duplicates.)

> each particular sequence has the same probability of selection as any other

Such a probability distribution, with every possible event having nonzero (positive) probability, is impossible for any except finite or countably−infinite sets. In the case of uncountably−infinite sets, probabilities of all but a finite or countably−infinite subset must be exactly zero, effectively reducing the distribution to a finite or countably−infinite probability space.

> is said that there exists a uniform probability distribution of the reals of the unit interval

Anyone who says that is wrong. The best that can be defined is a set of nested finite distributions, with increasing number of options at each nesting level. For example, it's possible to set up such a nested set of finite distrubitions such that the probability of nesting to within epsilon of any number is exactly 2/epsilon except at either endpoint where the probability is 1/epsilon.
Still the probability of hitting any point exactly is zero.
This is similar to the idea of *measure* (Lebesgue for example), where the measure of any single point or any countable subset is exactly zero, and the total measure of a set is *not* the sum of

all the single−point measures (it can be nonzero). Measure is necessarily additive (whole equals sum of parts) only for cases where the whole and each of the parts is a "measurable" set IIRC. For most theoretical subsets of the unit interval, the subset isn't "measurable" and there's no measure for it defined in the first place so you can't even assign meaning to the "whole equals sum of parts" if the whole or one of the parts is not measurable, IIRC.

> ... a random sample of the second binary digit (bit) of a real number's expansion is as well independently a sample of another real number's first digit.

Yes. Any uniform nested−finite−distribution system which is "uniform" has that property. This is a measure−theoretic, or equivalently metric−theoretic, space, not a probability space on the individual real numbers themselves. All you can say is that for any interval of length w, your "uniform" sequence generator will have probability w of ending up within that interval, probability 1−w of ending up outside that interval, and probability 0 of hanging forever on one endpoint of that interval so that we can never know by experiment that it goes in or out and in fact that it converges to exactly that endpoint. Open or closed intervals makes no difference, since the probability is zero that we have a problem, an undecideable inside point (endpoint of closed interval) or an undecideable outside point (endpoint of open interval).

> there is a bag (multiset) of i random real numbers

There's no such thing as random real numbers, except in the trivial case where all but a finite countably−infinite set are excluded from consideration, or where it's impossible to determine whether that's the case or not.

> In sampling one real number

It's not possible to sample one real number, except in the trivial case where all but a finite countably−infinite set are excluded from consideration (because you're using a determinstic algorithm to generate the nested intervals), or where it's impossible to determine whether that's the case or not (because you're using quantum mechanics and have no way to run an experiment forever and then come back to observe the entire result).

Suppose you had a magic genie that gave you random real numbers, uniformly distributed? The probability of any single number is

> zero, so applying a test of likelihood we have a bad result at the very first number and it only gets worse as one after another zero−probability event occurs. I can accept that zero−probability events might sometimes happen, after all they are *possible* even if very very unlikely. But if they happen over and over and over, I dismiss the genie as a fraud.
>
> But how is the genie supposed to establish that even one of the numbers presented isn't computable, hence selected from a known countable subset of the reals? If the genie admits she/he is producing only computable numbers, which have been manipulated to be uniformly distributed, I would have to go back to my original proof to refute that: There is no infinite sequence of identical positive terms that converges. So if you enumerate them (with the Turing oracle to decide which algorithms halt and which don't, and among those which don't halt which produce infinite digits and which hang after their last output), still you aren't consistent.
>
> The rest of what you say is nonsense, based on your false assumption that it's possible to sample actual real numbers, as opposed to merely intervals of arbitrarily small size.
>
> Summary:
> − If your sample space is finite, it can be uniform, p=1/n.
> − If your sample space is countably infinite, it can be all−nonzero, but it can't be uniform, it has to be some (absolutely−)convergent series, which implies terms arbitrarily close to zero as you go deeper down the sequence that are summed.
> − If your sample space is uncountably infinite, at best it can be countably−infinite nonzero, reducing to one of the previous cases.
> − If you have more than one sample space, such as nested intervals, or non−nested intervals with Cauchy bounds on diameters, you can get what's called a "distribution" or a "measure", but binning per each level of the nesting follows one of the first two cases above.

The probability distribution with an equal probability of each real in an interval being selected is well−known as the "continuous uniform probability distribution."

Re−read that: "an equal probability of (each real in an interval) being selected", where an interval is defined by its length instead of endpoints, locationless in parameterization. If by definition it's uniform (equal probability of each sample being an event) for even those degenerate (single−point, singleton, "arbitrarily small in size") intervals, that would be presupposition of such a beast, or simple illustrating it modelled by the limit of finite approximations, in a similar way as to how the unit impulse function, Dirac's delta, not a real function, yet with much utility, is described, besides how it is defined, in terms of its mathematical properties.

Consider for example http://www.itl.nist.gov/div898/handbook/eda/section3/eda3662.htm
where there is discussed a standard(−ized) continuous uniform
probability distribution.

The wikipedia entry defines that instead of each _element_ in the distribution's support set having being equally probable, that instead each _interval_ is equally probable, as an event. That's reasonable in pedantic standard measure−theoretic probability, where there's only countable additivity of probabilities to thus meet the definition of a probability distribution by, for example, its CDF. (EF is a CDF.)

http://en.wikipedia.org/wiki/Uniform_distribution_(continuous)

In sampling real numbers via "infinitely many samples" of fair coin flips, (http://www.google.com/search?q=%22infinitely+many+samples%22), basically any finite sample of length L indicates a sample of precision 1/2^L, then also independently samples of precision 1/2^{L−1}, ..., 1/2. In the limit, there are as described infinitely many samples, and then there is again the consideration that those samples with the least "randomness", eg unpredictable sequences given initial segments of the sequence, when sampled once are sampled many times.

That there can't be even countably infinite partitions of equal size of the unit interval corresponds to the result that then there would be an infinitesimal constant c such that the sum from one to infinity of c equals one. (Consider the integral of dx from zero to one, it equals one, and the Leibnizian style integral bar is a styled S for summation.) Were there such a beast, one of Cantor's bacilli (infinitesimals, Newton's first fluxion to the unit fluent, Leibniz' differential), then Vitali's proof would not hold and there wouldn't thusly be non−measurable sets. Where there's not that value, no definite integral actually "equals" the result that yields from analytic geometry in correspondence of the 2−D integral to area. (That is to say, for no finite difference is the integral generally correct, and for zero difference the sum of areas is constant zero.)

That half of the integers are even holds from that as a set, the integers is a set of numbers, and very basic number−theoretic results hold that the asymptotic density of the evens in the integers is one−half. The set of numbers has all the numbers' baggage.

I'm not a troll, I only argue mathematically what I see as true.

EF is a CDF (monotonic (strictly) increasing, greater than or equal to zero and less than or equal to one, and in the limit one), of a uniform probability distribution over the naturals, which I've described in construction in ZF.

The sum for natural n from one to infinity of 1/2^n equals one. Perhaps that's not for all an agreeably formalist perspective, but it is a very realist one.

Ross

−−
Finlayson Consulting

.