Re: Group of transformations

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On Tue, 13 Nov 2007 09:06:52 EST, jane <jane1806@xxxxxxxxxx> wrote:

> Suppose i have G a group of transformations on R^2, generated by the translations by integers Z^2 and rotation by 90 degrees around a fixed point O. What is the quotient of R^2 / G by the action of this group ?

This question calls strongly for a presentation with complex numbers using the standard bijection (x,y) |−> x+i*y. Then your translations are translations in |C by so−called Gaussian integers (m+n*i for m,n ordinary integers) and your rotation becomes multiplying by i. Your group G is then easily seen as the semi−direct product of its normal subgroup T made of said translations and the cyclic subgroup generated by said rotation. Each element of G has as unique form z |−> (i^k)*z+c where k is 0,1,2 or 3 and c is a Gaussian integer.

Now it's not quite clear what you are exactly asking for, because there is the trivial answer: your quotient is exactly what the definition says it is ... My idea − also out of curiosity for myself − was to try to get a 'feeling' for what's happening here. First one realizes following facts.

Let S be the closed square limited by the real and imaginary axes and their parallels through 1/2 * (1+i) ... in other words made of the numbers x+i*y for x,y both in the real interval [0,1/2]. The images of S by all elements of G are squares with sides parallel to the axes (like S itself) covering the complex plane and the images of the interior of S (the open square with the same vertices) are all disjoint. (The said images of S are also obtained by applying the translations by half Gaussian integers to S.) So your quotient is topologically the same as the quotient of S obtained by glueing somehow the sides of S. The quesion is then: exactly how? in other words: which points of the border of S are congruent modulo G? My answer − which I did not verify completely − is: a) the sides containing 0, by z <−> i*z, b) the other two sides by z <−> i*z+1 and nothing else. Both a) and b) include glueing 1 with i, and they in fact do the same as the symmetry with resp. to the diagonal

Re(z) = Im(z) of S (for the sides of S). So we just glue points of the
sides of S that are symmetric with respect to said diagonal.

If this is correct, then your quotient is topologically a sphere
which is also obtainable as a quotient of the torus R^2 / T
by the action of the order 4 cyclic group generated by transformation
of the torus derived from z |−> i*z

BTW your G looks like a 2D crystallographic group or a subgroup of
such. (It is said that 2D and 3D crystallographic groups have been
completely classified and that it has been proven that there are
essentially only a small finite number of them for each dim.)
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