Re: Positive/Negative after taking the square root

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matt271829−news@xxxxxxxxxxx wrote:

> On Nov 29, 12:06 am, matt271829−n...@xxxxxxxxxxx wrote:
>
> <snip>
>
>> When you substitute u = x^2 you're losing information about which of the two possible values of x you're dealing with. (This always happens when the inverse function is multi−valued.)
>
> A simple example of this, which I suppose is vaguely related to the matter in hand, would be a definite integral between x = −1 and x = 1. With the substitution u = x^2 this appears to become an integral between u = (−1)^2 = 1 and u = 1^2 = 1, which is obviously wrong.

Of course that wouldn't normally happen in practice. (After all, who would really choose to integrate x^2 using a substitution.) But there are practical cases in which this sort of thing happens:

Suppose we wish to calculate the average distance of a planet from a star, about which the planet is in an elliptical orbit. (The average is to be taken with respect to the angle t between the major axis of the ellipse and the line joining the star and the planet.) An abstract problem of exactly the same form is

Given the ellipse r = 1/(2 + cos(t)), expressed in polar coordinates, find the average of r wrt t around the ellipse.

Of course, r_avg = 1/(2pi) integral(t=0 to t=2pi, 1/(2 + cos(t)) dt).

To evaluate that, students typically think of one of two methods.

Method 1:
Referring to the integral tables at the back of the text or using a CAS like Mathematica, get an antiderivative; then just use the FTC.

Regardless of whether tables or CASs are used, one most often finds,
for a > b,

integral(1/(a + b cos(x)) dx) =
2/sqrt(a^2 − b^2) arctan(sqrt(a^2 − b^2)/(a + b) tan(x/2))

stated _without any restriction on x_.

So in our problem, an antiderivative would seem to be

F(t) = 2/sqrt(3) arctan(tan(t/2)/sqrt(3))

But then using FTC, we get F(2 pi) − F(0) = 0 − 0 = 0 and so r_avg = 0, meaning that our planet would be _very_ hot, or more likely, that something went wrong during our calculation.

Method 2 (for the industrious student who eschews tables or CASs):
Use the Weierstrass substitution u = tan(x/2).

If the student uses this substitution to evaluate an _indefinite_ integral, the outcome will be the same as in Method 1. OTOH, if the student uses this substitution to evaluate the definite integral, then things are immediately seen to be wrong because the limits of integration, t=0 to t=2 pi, become u=0 to u=0.

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As many readers are already aware, the simple thing to do is to get the average distance over just _half_ of the orbit. That, by symmetry, will be the same as the average over the whole orbit.
With F(t) = 2/sqrt(3) arctan(tan(t/2)/sqrt(3)),

r_avg = 1/pi integral(t=0 to t=pi, 1/(2 + cos(t)) dt) =

1/pi (lim_{t −> pi−}F(t) − F(0)) = 1/pi (2/sqrt(3) pi/2 − 0) = 1/sqrt(3).

But to get the correct answer, of course we should not be _forced_ to use symmetry! Since 1/(2 + cos(t)) is continuous on the whole real line, it has an antiderivative which is valid on the whole real line. The tables or CASs actually should have given, for a > b, G(x) =

(x − 2 arctan(b sin(x)/(b cos(x) + a + sqrt(a^2−b^2))))/sqrt(a^2−b^2)

or something equivalent, if they wanted to give an antiderivative valid on the whole real line. Then we could use FTC without having to resort to symmetry:

r_avg = 1/(2pi) integral(t=0 to t=2pi, 1/(2 + cos(t)) dt)

= 1/(2pi) (G(2pi) − G(0)) = 1/(2pi) (2pi/sqrt(3) − 0) = 1/sqrt(3).

David
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