The Law of the Excluded Middle again (long)

Source: http://sci.tech−archive.net/Archive/sci.math/2007−12/msg00180.html

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- *Date*: Sat, 01 Dec 2007 18:38:44 +0000

If this post seems odd, please think of it as being about a "dog that didn't bark in the night", and the reason for it will become clear.

A few weeks ago, in the thread "A quote (and question) about intuitionism" (which I have temporarily let drop, with the ball still in my court) I said I had intended to pick some simple everyday example of a mathematical argument in which the Law of the Excluded Middle is used.

Such a simple everyday argument has just presented itself.

A recent thread asked for a "slick" proof that x^y + y^x > 1 for all positive real numbers x and y; and a reference was given to a proof of this by Wade Ramey in sci.math in 2003. I wondered about an extension to more than two variables (I see this was also asked in the earlier thread), and found a simple deduction of the case of 3 variables from the known case of 2 variables. I posted about this, and Wade gave a neat solution, similar to my own.

Absolutely no deep ideas were involved in any of this, and it was little more than a matter of idle curiosity (at least on my part). At most, when I was about to drop the matter out of a lack of real interest, I piqued my interest again by saying to myself, silently, "Well, the infimum of the set of numbers x^y + y^z + z^x certainly /exists/, even though I have no idea how to /construct/ it", and this made me curious enough to see if I could form any idea of what the number actually was. (It soon enough turned out to be 1.)

Again, nothing remarkable about this. But that is just what is wanted: a simple, unremarkable example of everyday mathematical reasoning in action, using the Law of the Excluded Middle. We should be able to use it to examine the role of the LEM, without distraction (caused either by complications in the mathematics, or by an absence of any real bread−and−butter mathematics to think about).

In the present case, the LEM is used in at least two places.
(That is, explicitly; probably it is used implicitly as well.)

We first argue that if any of x, y, or z is >= 1, then the sum
x^y + y^z + z^x is > 1. So we say that it suffices to consider
only the case where all of x, y and z are in (0, 1) (although in
fact the argument more naturally considers them all to be in the
half−open interval (0, 1]).

Either way, we are appealing to the LEM, by saying "either
(x >= 1 or y >= 1 or z >= 1) or (x < 1 and y < 1 and z < 1)"
(and then deducing the same conclusion in both cases).

We then make a further division into cases, say (as in Wade's
argument) into the cases x <= y and x >= y. (As before, there is
no need to use a sharper and less symmetrical alternative x > y.)

If x <= y, then z^x >= z^y, therefore y^z + z^x >= y^z + z^y. If
x >= y, then ... actually, I think the more complicated argument
that I used actually is needed at this point! (I assumed, without
examining it closely, that I had complicated things unnecessarily.)

Unfortunately this now become a less childishly simple example
than I wanted it to be, but it is still elementary, and far from
complicated. (Also, it is still realistic. Or rather, it is real,
in the sense that it was used in practice, not just concocted for
philosophical purposes.)

One way to phrase the argument is "either 0 < x <= y <= z <= 1
or 0 < z <= y <= x <= 1 or ..." (four other cases, which can
then be reduced to these two "by symmetry", i.e. by renaming
the variables).

(Again the argument most naturally uses overlapping cases, but
again this doesn't affect the essential point at issue.)

If 0 < x <= y <= z <= 1, we can argue y^z + z^x >= x^z + z^x,
or y^z + z^x >= y^z + z^y. Either way, we can use the known
result x^y + y^x > 1 (with renaming of variables) to deduce
that, in this case, x^y + y^z + z^x > 1.

If 0 < z <= y <= x <= 1, we can argue x^y + y^z >= x^y + y^x,
or x^y + y^z >= z^y + y^z, and deduce the same conclusion in
this case as well.

(I am spelling this out laboriously in order to avoid making
any slip like the one that I, and apparently also Wade, made
earlier.)

The point I am making with this post might either turn out to
be something objective, or merely something about a lack in

myself as a student; I am not yet in a position to say which it is; but in either case (am I appealing to the LEM here as well?!), it is worth spelling it out (at least so that I can learn something).

I pride myself on having a reasonably well−functioning bullshit detector, and the point is that (like "the dog that didn't bark in the night") it didn't give the slightest peep, all the time I was working on this problem, reading Wade's solution, and seeing what I could learn by doing a "post mortem" on the problem. It was only after it was all over (I thought!) that the point about the Law of the Excluded Middle occurred to me.

Can someone explain to me why my bullshit detector /should/ have emitted a loud warning (about the use of the LEM in this argument), and perhaps how I could train/tune my bullshit detector to work better in the future? Or, is it possible that there is simply nothing wrong with the argument? That is what I still believe, but I am open to correction.

That last question raises another (quite distinct) point (which has also come up in recent discussions about the philosophy of mathematics in sci.math): of course it /is/ still perfectly possible that there is something wrong with the argument!

I think we have already had an example of this, when Wade gave a version of the argument that was simpler than mine, and I just assumed (and so, I guess, did Wade) that it was equally valid, and that my argument had been overcomplicated. Nevertheless we were both probably "right" to assume that there was nothing essentially wrong with it, and that the problem was solved.

So what do I mean when I say, so confidently, that there is "nothing wrong" with the argument (and even that there was nothing "essentially" wrong with it before, even though I now think that there /was/ an error)? Am I (as galathaea might perhaps suggest) claiming something like Papal infallibility? Am I about to start spluttering about the need to crucify the vandals who dare lay siege to Rome? Am I a "fundamentalist"? Am I even claiming to be "omniscient"?

Hardly! Please believe me when I say that yesterday (and indeed today) I could hardly have been in a less confident or self−aggrandising mood; all I did was cheer myself up in a depression by working at a simple and unimportant bit of mathematics. I'm now making a song and dance about this near−triviality because of the existence of philosophies of mathematics that would drain all meaning from this sort of activity (so that it wouldn't cheer me up for long, if at all).

I mean it is my (fallible, uncertain) opinion that the argument

does not contain any logical fallacy. (In particular, I don't think that the Law of the Excluded Middle, when applied to mathematics, is a logical fallacy.) As a background to this, the belief that an argument is valid causes, psychologically, a sense of certainty.

Also, again as a background, there is a sense that the question of the validity or invalidity of arguments (expressed initially in informal terms, as is the example I have given) can be expressed as a question as to whether the argument does or does not follow certain "rules", which can be described. I seriously have my doubts about this (but I admit at once that this may be just ignorance on my part), just as I do not believe at all that questions of morality reduce to obeying or following rules (although some vague notion of universalisability surely does have some importance, and I must admit I have scarcely begun to examine this, and haven't read Kant, etc.).

I'm sure much could be written about these general background questions of psychology and philosophy (and I would be extremely interested), but I don't want to digress too far, for the moment, from the question of the validity of the "Law" of the Excluded Middle. (The scare quotes are to remind us of these background connections with other questions.)

It's not that I don't see all sorts of difficulties in the foundations of mathematics. When I was an undergraduate student (so long ago that I can scarcely believe it ever happened), I felt uneasy about lots of things, but could see no way of discussing them or otherwise coming to grips with them, so I just tried to deal with them by ignoring them.

And indeed, there is wisdom in that approach. (Cauchy: "Carry on, and faith will come." − Can someone authenticate this quotation?) I have been doing something like it for the last three years: studying maths in a quiet, mainstream way, and trying (when I am subscribed here) not to be /too/ distracted by all the foundational threads that so often erupt in sci.math. All my difficulties will have to be dealt with eventually, but I am in no hurry to do it, knowing the difficulty of the task.

It's just that worrying about the validity of the Law of the Excluded Middle has /never/ been among my difficulties! (Not in pure maths, that is. In applied maths, it is a quite different story, but applied maths is a minefield for me, and this difficulty is only one of many.)

What has become a difficulty now is that the Law of the Excluded Middle is seriously worried about by people whom I respect (partly because they know vastly more maths than I do, and are probably more talented as well), and because I am now trying to learn how other mathematicians think, I am faced with trying to learn to have the same doubts and see the same problems as these people see. And I am coming up blank.

This won't stop me trying to learn mathematics; nor will it stop me "believing" in the validity of a simple and elementary proof like the

one I have just been labouring over (perhaps very annoyingly, I can't tell); but it is now a genuine problem in my path (along with genuine mathematical problems themselves).

Just as I don't think "crank" posters should always be vilified for what seems like (and may be) arrogance, I hope I won't be vilified by disbelievers in the LEM for what seems like (and for all I know may be) a presumption of omniscience, or an unthinking dogmatism. I am saying what I think − believing, of course, that what I think is true, is true! − but not in order to shove it down anybody's throat, and not to protect it from criticism, but rather to open it up to criticism, and thus either to make it stronger (if it deserves to be), or else to abandon it (if it deserves /that/ fate, instead).

I think Bertrand Russell said something like, "The virtue of a logical argument is not that it compels certainty, but that it suggests doubt." (Again, can someone find the exact quotation for me?) I want what I write to be taken in something like that spirit. That is, I am saying, "This is what I believe. I think it is true and important. Moreover, it matters a lot to me personally. Can you show me where I am wrong, or is it possible that I am right?"

−−

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(twirlip@ eats spam; reply to angusrod@)
Contains mild peril

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