Re: Example of a set requiring d+1 points (Caratheodory theorem)

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- *From*: Kevin Buhr <buhr+un@xxxxxxxxxxx>
- *Date*: Sat, 23 Feb 2008 02:22:18 −0600

Kevin Buhr <buhr+un@xxxxxxxxxxx> writes:

> For an example of a *connected* A where you still need d+1 points, I don't believe you'll find one in R^1 or R^2, but you'll find it easy to find one in R^3. Let A be the union of three line segments joined at a single point (so their convex hull is a tetrahedron). Note that no interior point can be expressed as a linear combination of only two points of A.

Oops.. Well, now that I reread what I wrote, that's hardly going to work, is it? In R^3, you obviously want an example where *four* points are required.

And, now that I've thought about it, I think in R^d for a connected set A, you'll never need all d+1 points. You can use connectedness to "eliminate" a point as follows.

Suppose you have an x in A (subset of R^d) expressed as a convex combination (c.c.) of d+1 points in A: x_0,x_1,...,x_d. If these points aren't all linearly independent, you can obviously express x as c.c. of a subset of d or fewer of them, and you're done, so suppose they *are* independent. Then, the convex closure of the points is a d−simplex S(x_0,x_1,x_2,...,x_d). If x is on the boundary, again, you can immediately express x as a c.c of a subset of d or fewer of the points, so suppose x is in the interior of the simplex.

Now, as A is connected, there is a path from x_0 to x_1. Consider the "shrinking" d−simplex S(y,x_1,x_2,...,x_d) as y moves along the path from x_0 to x_1. Geometrically, "x" is contained in the interior of the original simplex, and as y goes from x_0 to x_1, the simplex "shrinks". At some point, either at y=x_1 or at some earlier time on the path, the simplex becomes "flat"−−−it's no longer a d−simplex. It's either a (d−1)−simplex or−−−if this happens before y=x_1 and depending on the shape of the path−−−maybe the union of two (d−1)−simplexes. But sometime between y=x_0 and the first "collapse" of the simplex, the boundary of the shrinking simplex (or its collapsed version) must contact x. At that point, x can be expressed as a c.c. of y and d−1 of the points (x_1,x_2,...,x_d).

Visualizing this in R^2 is easiest: if x is expressed as a c.c. of three linearly independent points, it's in the interior of the triangle formed by those three points S(x_0,x_1,