Re: A Formula for Pi

Source: http://sci.tech−archive.net/Archive/sci.math/2008−06/msg02281.html

- *From*: rob@xxxxxxxxxxxxxx (Rob Johnson)
- *Date*: Mon, 23 Jun 2008 10:50:40 GMT

In article <ef3600bd−5d9c−41d4−9442−f4703d45ba38@xxxxxxxxxxxxxxxxxxxxxxxxxx>,
Mensanator <mensanator@xxxxxxx> wrote:

> On Jun 22, 7:29 pm, r...@xxxxxxxxxxxxxx (Rob Johnson) wrote:
>
>> In article
>> <2e23fae6−a30d−49d8−9520−62d16cecc...@xxxxxxxxxxxxxxxxxxxxxxxxxxxx>,
>> Mensanator <mensana...@xxxxxxx> wrote:
>>
>>> On Jun 22, 8:48 am, r...@xxxxxxxxxxxxxx (Rob Johnson)
>>> wrote:
>>>
>>>> In article
>>>> <rubrum−29BD9A.21370621062...@xxxxxxxxxxxxxxxxxxxxx>,
>>>> Michael Press <rub...@xxxxxxxxxxx>
>>>> wrote:
>>>>
>>>>> In article
>>>>> <41c3a2f3−d399−42f6−b21d−ce897b6ba...@xxxxxxxxxxxxxxxxxxxxxxxxxx
>>>>> Mensanator
>>>>> <mensana...@xxxxxxx>
>>>>> wrote:
>>>>>
>>>>>> On Jun 20,
>>>>>> 2:35 pm,
>>>>>> r...@xxxxxxxxxxxxxx
>>>>>> (Rob
>>>>>> Johnson)
>>>>>> wrote:
>>>>>>
>>>>>>> In
>>>>>>> article
>>>>>>> <4f348778−0c41−45e4−bb51−9fcc3dca9...@xxxxxxxxxxxx
>>>>>>> Mensanator
>>>>>>> <mensana...@xxxxxxx>
>>>>>>> wrote:
>>>>>>>
>>>>>>>> On
>>>>>>>> Jun

20, 10:57 am, Maury Barbato <mauriziobarb...@xxxxxxxx> wrote:

> Jose Carlos Santos wrote:
>
> > On 20−06−2008 7:16, Maury Barbato wrote:
> >
> > > I found in the book "The Penguin Dictionary of Curious and Interesting Numbers" by Wells the following formula involving pi
> > >
> > > (pi − 3)/4

= sum_{k=1 to infty}[(−1)^(k+1)]/[2k*(2k+

Is there anybody who knows a proof of this

wonderful

series?

It's a very very beautiful formula!

Be that as it may, how fast does it converge? How many terms do I have to sum to get 100 decimal place accuracy?

He never claimed that it was an efficient way to compute pi,

I didn't say he claimed it was efficient. I'm _asking_ whether or not it is efficient.

simply that it was a beautiful formula.

Fine. It's beautiful. No disagreemrnt there.

It
is
a
personal
opinion,

Which
everyone is
entitled to.

but
I
agree.

oo
−−−
k+1
1
>
(−1)
−−−−−−−−−−−−−−
−−−
2k(2k+1)(2k+2)
k=1

oo
1
−−−
k−1
1
2
1
=
−
>
(−1)
(
−−
−
−−−−
+

−−−−
)
[partial
fractions]
2
−−−
2k
2k+1
2k+2
k=1

oo
1
−−−
k
1
=
−
+
>
(−1)
−−−−
[collapse
telescoping
terms]
4
−−−
2k+1
k=1

1
pi
=
−
+
(
−−
−
1
)
[Gregory
series]
4
4

pi − 3 = −−−−−− 4

Being an alternating series with monotonically decreasing terms, the error is less than 1/(8n^3) after n terms.

Some of us don't know how to do this. But given the series, I can write a program to apply it.

But I would prefer a series that converges in ~300 terms to the one that converges in ~10**34 terms.

How about Ramanujan's
series that gets eight
decimal places for each
term?

1 2.sqrt{2} (4n)! [1103 +
26390n]
−− = −−−−−−−−− sum_n
−−−−−−−
−−−−−−−−−−−−−−−
pi 9801 (n!)^4 (4*99)^4n

Also look up Euler's (who
else?) transformation for
accelerating
alternating series. In essence
a difference table.
Abramowitz and Stegun,
3.6.27.

<http://mathworld.wolfram.com/EulersSeriesTransformation.html>

At
<http://www.whim.org/nebula/math/eulerxform.html>,
there is a
rigorous proof of the Euler Series
Transform.

However, if we apply the acceleration using
the identity

−−− j C(m,j) k 1
> (−1) −−−−−−−− = −−− −−−−−−−−−−
−−− C(n+j,k) k+m C(n+m,k+m)
j

proven at
<http://www.whim.org/nebula/math/binom.html>,
we get

m
−−− j C(m,j)
> (−1) −−−−−−
−−− 1/2+j
j=0

2^{2m+1} m! m!
= −−−−−−−−−−−−−−
(2m+1)!

Thus, the Euler Series Transform says

pi

oo
−−− 2^{m+1} m! m!
= > −−−−−−−−−−−−−
−−− (2m+1)!
m=0

Which converges much faster than the
Gregory series, better than a
ratio of 2 per term.

This was even easier to implement.

Did you take advantage of the fact that the ratio of one term to the next is m/(2m+1). That makes it a snap to implement:

1 1 2 1 2 3 1 2 3 4
pi = 2 (1 + − + − − + − − − + − − − − + ...)
3 3 5 3 5 7 3 5 7 9

> Uh...that's the original formula I used. I thought
> I mentioned that somewhere. No wonder I got virtually
> the same number of terms.

In another thread on this topic, I found your reference:
<d835b1c3−d8b5−4e82−917e−804e7c681ed9@xxxxxxxxxxxxxxxxxxxxxxxxxxxxx>
I missed it since it was in another thread.

In any case, this is the series that comes from the application of
the Euler Series Transform to the Gregory Series. Interesting on its
own.

However...

pi time: 0.0 seconds
31415926535897932384626433832795028841971693993751058209749445923078164062
862089986280348253421170679821480865132823066470938446095505822317253594081
284811174502841027019385211055596446229489549303819644288109756659334461284
756482337867831652712019091456485669234603486104543266482133936072602491412
737245870066063155881748815209209628292540917153643678925903600113305305488
2046652138414695194151160943

pi_also terms: 1328 time: 0.344000101089 seconds

31415926535897932384626433832795028841971693993751058209749445923078164062
862089986280348253421170679821480865132823066470938446095505822317253594081
284811174502841027019385211055596446229489549303819644288109756659334461284
756482337867831652712019091456485669234603486104543266482133936072602491412
737245870066063155881748815209209628292540917153643678925903600113305305488
2046652138414695194151160942

pi == pi_also: True

>>> arctan(1/5) terms: 287 arctan(1/239) terms: 85 time:
>>> 0.0780000686646 seconds
>>>
>>> 31415926535897932384626433832795028841971693993751058209749445923078164062
>>> 86208998628034825342117067982148086513282306647093844609550582231725359408 1
>>> 28481117450284102701938521105559644622948954930381964428810975665933446128 4
>>> 75648233786783165271201909145648566923460348610454326648213393607260249141 2
>>> 737245870066063155881748815209209628292540917153643678925903600113305305488
>>> 2046652138414695194151160943
>>>
>>> pi == pi_Machin: True
>>>
>>> Euler terms: 1329 time: 3.0 seconds
>>>
>>> 31415926535897932384626433832795028841971693993751058209749445923078164062
>>> 86208998628034825342117067982148086513282306647093844609550582231725359408 1
>>> 28481117450284102701938521105559644622948954930381964428810975665933446128 4
>>> 75648233786783165271201909145648566923460348610454326648213393607260249141 2
>>> 737245870066063155881748815209209628292540917153643678925903600113305305488
>>> 2046652138414695194151160942
>>>
>>> pi == pi_euler: True
>>>
>>> It took just about as many terms as my original formula
>>> and is much slower to calculate. And my original can't
>>> hold a candle to the Machin formula.

>> I hope you did not use the factorial function in your implementation,

> Well, a bit. But I certainly didn't re−calculate m! every term.
> Obviously, not quite as efficient as the original, so there's
> some needless calculation in my Euler algorithm.

That would explain the 3 second execution time.

but used the term to term ratio I pointed out above. If so, it should be a little over

"Starnge, the line went dead just as Professor Press was about to reveal the secret formula."

Okay, I'm confused, but that is not unusual.

I wasn't trying to get the quickest converging series, I was trying to show that Euler Series Accleration can take a terribly slowly converging series, like the Gregory Series, and make it into a fairly quickly converging series.

Well, I'm not trying to find the quickest either, just an easy to implement that does a reasonable job. I'm not trying to do a billion digits or anything (this algorithm isn't up to it).

The series that I accelerated from the Gregory series has only a term to term ratio of about 2, which yields about .3 digits per term (about 1335 terms for 402 digits).

The arccot(5) part of the Machin formula has a ratio of about 25, and yields about 1.4 digits per term; the arccot(239) part has a ratio of about 57121, and yields more than 4.75 digits per term. This comes out to about 1.08 digit per term altogether (about 372 terms for 402 digits).

The quickest converging Machin−like series I have seen is Gauss's

pi = 48 arccot(18) + 32 arccot(57) − 20 arccot(239)

which yields about 1.12 digits per term altogether (about 359 terms for 402 digits). However, I just looked up on MathWorld, and there it has

pi = 732 arccot(239) + 128 arccot(1023) − 272 arccot(5832)

+ 48 arccot(110443) −48 arccot(4841182) − 400 arccot(6826318)

which comes in at over 1.51 digits per term.

Thanks for the info, I may try one of these just for laughs.

> I'm certainly less concerned about the difference
> between 372 and 359 than between 372 and 10^34.

Understandable.

Rob Johnson <rob@xxxxxxxxxxxxxx>
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