Re: tetration and logaritms

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On Jul 9, 11:59 am, amy666 <tommy1...@xxxxxxxxxxx> wrote:

> we all know the taylor series for log(1+x) which holds for all real x between −1 and 1.
> this series starts with 0 + x − x^2/2 + ...
> and from looking at the first terms of this taylor series we see that we can get series reversion which of course gives exp(x) − 1.
> fractional iterations can be done for exp(x)−1 since its a taylor series with f(0) = 0. (and its unique)
> and thus also for log(1+x) , either directly since log(1) = 0 or from the series reversion of the fractional iterations of exp(x)−1.

Hmmm, this looks interesting.

Let us return to the U−tetration notation first introduced by Gottfried Helms. In this notation:

exp(x)−1 = U_e(x,1)

I see no reason why we can't write:

log(1+x) = U_e(x,−1)

and start using negative values for the second parameter of U_e (which Gottfried calls "h" for "height").

Then tommy1729 is really asking for U_e(x,−1/2). So now I wonder whether the same matrix that Gottfried would use to find, say, U_e(x,1/2) would work for U_e(x,−1/2) as well.

I don't have Gottfried's full matrix method available (he's explained it but it's a bit too complex to follow), but here's a simpler version that only works for base e. (This is because Gottfried has discovered that the Taylor coefficients are a bivariate polynomial that evaluates to 0/0 when the base is either e or 1/e. But L'Hopital's Rule can be used in this case to give a polynomial in the single variable h. This trick simply gives us this polynomial in h.)

Suppose we only want the first five terms of our Taylor series for U_e(x,h). So we begin with a 6x6 matrix with zeros everywhere, except on the superdiagonal where it is one. This matrix corresponds to U_e(x,0) = x, the identity function.

To find the matrix for U_e(x,1), we take the Taylor series for exp(x)−1 (which is x+x^2/2+x^3/6+x^4/24+x^5/120), plug in the previous matrix for x. Naturally, this gives us a strictly upper triangular matrix with 1 on the superdiagonal, then 1/2 on the diagonal above that, then 1/6 on the next diagonal, and finally 1/24 and 1/120 in the upper−left corner. This tells us that U_e(x,1) is x+x^2/2+x^3/6+x^4/24+x^5/120 (but this is trivial).

We then repeat the process by plugging in the new matrix for x in the series for exp(x)−1, to find U_e(x,2). Here we find that not only does the superdiagonal contain 1, but so does the diagonal above it. The next diagonal has the value 5/6, and finally 5/8 and 13/30 in the upper−left corner. So we have that U_e(x,2) = x+x^2+5x^3/6+5x^4/8+13x^5/120.

Continuing in this manner, we find:
U_e(x,3) = x+3x^2/2+2x^3+5x^4/2+179x^5/60
U_e(x,4) = x+2x^2+11x^3/3+77x^4/12+163x^5/15

We could continue like this forever, but this is all we need to find the polynomials in h. We simply take the coefficients for the x, x^2, x^3, ..., terms, view them as functions of h, and then take the Lagrange interpolating polynomial.

The x terms are trivial: 1, 1, 1, 1, 1. So we see that the linear coefficient is always 1.

The x^2 terms is also easy. Starting with U_e(x,0) they form the patten 0, 1/2, 1, 1/2, 2. So the x^2 coefficients must be h/2.

For x^3 we have 0, 1/6, 5/6, 2, 11/3. The x^3 coefficients work out to be h^2/4−h/12.

For x^4 we have 0, 1/24, 5/8, 5/2, 77/12. The x^4 coefficients work out to be h^3/8−5h^2/48+h/48.

For x^5 we have 0, 1/120, 13/30, 179/60, 163/15. The x^5 work out to be h^4/64−13h^3/144−h^2/24−h/180.

Notice that the polynomial for x^n always works out to be a polynomial in h of degree n−1.

Putting this together, we have the Taylor polynomial
U_e(x,h) = x+(h/2)x^2+(h^2/4−h/12)x^3+(h^3/8−5h^2/48+h/48)x^4
+(h^4/64−13h^3/144−h^2/24−h/180)x^5.

And this seems to work for any _positive_ value of h that
Gottfried chose. So for h = 1/2:

U_e(x,1/2) = x+x^2/4+x^3/48+x^5/3840

And so there's nothing stopping us from letting h be negative:

U_e(x,−1) = x−x^2/2+x^3/3−x^4/4+x^5/5

And this of course is the well−known Taylor series for log(1+x)
that tommy1729 already mentioned in his post. Finally, h=−1/2:

U_e(x,−1/2) = x−x^2/4+5x^3/48−5x^4/96+109x^5/3840

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