Approaching the infinite binary tree

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- *From*: LudovicoVan <julio@xxxxxxxxxxxxx>

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I am thinking along the following lines, and I'd greatly appreciate some feedback.

Given the set of the extended naturals: N* := N u {w}, where 'w' (omega) is the limit ordinal.

Let's consider the infinite binary tree, that is −informally− a binary tree with w levels. We can represent, in bilogarithmic scale, this tree embedded in half of a unit square, as follows:

```
k = 0 _ _ 0
/\
/ \
/ \
1 _ / \ _ 1
/\ /\
2 _ / \ / \ _ 2
... /\ /\ /\ /\ ...
w _ /_______________\ _ w
```

Every binary sequence can be associated to a specific path in this tree by the following sequential (in fact, inductive) rule over the levels of the tree:

−− Base case: start from point (0,0) at level 0, written (0;0)_[0]

−− Successor case: from point (m,n)_[k−1],
go to point (m + 1/[2^(−k)] , n)_[k]
if the next (the k−th, counting from 1) digit
in the sequence is "0",
go to point (m , n + 1/[2^(−k)])_[k]
otherwise.

We are, equivalently, embedding the infinite binary tree into a half of a square lattice with −informally− 2^w points per side, normalized to the unit square. Other mappings were possible, but this is irrelevant up to constant factors.

Here the limit point for any given sequence provably exists and is unique(*), it corresponds to a point on the limit diagonal: specifically, to one of the −informally− 2^w points that make up the limit diagonal in our lattice. (In fact, we have used *finite* induction, which works for all finite though *unbounded* processes. We'll need *transfinite* induction to get "all w levels" and properly talk about *infinite* strings, or lists.)

In any case, a non−standard result seems already here: the uniqueness of our limits implies a *bijection* between sequences and limit points. To each binary sequence corresponds one and one only limit point. So, for instance, here it is *not* correct to take the sequence "0(1)" to be equal to (to correspond to the same limit point as) "1(0)", there are no such equalities. Indeed, even just geometrically, from the diagram above we can see that *all* distances get halved at each step, so that the distances between the points on the limit diagonal tend *uniformly* to zero, and all points remain uniformly distributed and distinct.

I need solid ground to proceed further, so I'd appreciate some feedback as to the soundness of the construction and the (few, basic) results given here.

Thank you,

−LV

(*) If we consider our construction as providing the points over successive diagonals in the given lattice (the dotted lines in the diagram below), we have an iterated function system (IFS) that provably converges to a specific attractor, here the set of points on the limit diagonal.

```
k = 0 _ _ 0
/\
/ \
/ \
1 _ /......\ _ 1
/\ /\
2 _ /..\..../..\ _ 2
/\ /\ /\ /\
```

The number of points on a diagonal at step k is 2^k, and they delimit (2^k)−1 equal intervals, each of (normalized) length equal to 1/[(2^k)−1]. In the limit case we can −informally− say that we have 2^w limit points delimiting (2^w)−1 equal intervals, each of length 1/[(2^w)−1]. With the theory of IFS we can prove (I won't try) that a distance function expressed in these geometrical terms provides a *complete* metric space.
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