Re: Question on algebraic numbers

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- *From*: Bill Dubuque <wgd@xxxxxxxxxxxxxxxxxxxx>
- *Date*: 02 Feb 2009 03:54:34 −0500

dmr5713@xxxxxxxxx wrote:

> The algebraic closure of Q (the rationals) is, of course, formed by adjoining to Q the roots of all polynomials over Q.
>
> Consider now the field consisting of all numbers that can be written as finite expressions involving addition, subtraction, multiplication, and division of integers, raising to rational powers, and compositions of these operations. This is obviously a subfield of the algebraic closure of Q.
>
> Question: is it a proper subfield? The fact that one cannot write any algebraic formula for the roots of polynomials over Q of degree greater than or equal to five does not mean that such roots cannot be written as some finite expressions of integers. (To put that differently, the insolvability of a polynomial over Q does not mean that its roots may not happen to be expressible as finite expressions involving integers that bear no generalizable algebraic relation to the coefficients of the polynomial.)
>
> In other words, are there algebraic numbers that not only cannot be written as general finite expressions in the coefficients of polynomials that may define them, but that cannot be written at all as finite expressions involving integers, the arithmetic operations, and raising to rational powers?

Although your question has already been answered, below I add some further related remarks that I suspect may be of interest. First, below is appended an excerpt from Franz Lemmermeyer's 12: The failure of unique factorization.

As a matter of fact, however, E. Heine defined algebraic integers in [Hei] as numbers that can be constructed from the rational integers by addition, multiplication, and raising to m/n−th powers with m,n positive integers. He then goes on to show that every such number is integral in the modern sense, i.e. it is a root of a monic polynomial with integral coefficients. The converse is, of course, false, as

Heine's construction gives only algebraic numbers that are solvable,
i.e., that can be expressed in terms of radicals, and Heine's claim
that any solvable root of a monic polynomial with integral coefficients
is integral in his sense is still an open problem (see Exercise 12.17)
since his proof is not valid.

12.17 Let L be a number field such that its normal closure N has a
solvable Galois group. Is it true that the ring of integers O_L
coincides with the subset of L consisting of algebraic integers in
Heine's sense?

[Hei] E. Heine, Der Eisensteinsche Satz uber die Reihenntwickelung
algebraischer Functionen, J. Reine Angew. Math. 45 (1853), 285−302

Second, the buzzword you're looking for is root−closure / root−closed,
e.g. see the reviews below.

−−Bill Dubuque

92f:13006 13B22
Anderson, David F.(1−TN); Dobbs, David E.(1−TN); Roitman, Moshe(IL−HAIF)
Root closure in commutative rings.
Ann. Sci. Univ. Clermont−Ferrand II Math. No. 26 (1990), 1−−11.

Let S be a subset of N. If A < B are rings, A is said to be S−root closed in B
if, whenever b in B and b^n in A for some n in S , then b in A . The smallest
subring of B which contains A and is S−root closed in B is called the total S
−root closure of A in B and is denoted by R^S_oo(A,B) . It is shown that
R^S_oo(A,B) = \/ {R^S_m(A,B) : 0 <= m <= oo}, where R^S_0(A,B) = A and,
for m>0, R^S_m(A,B) is the subring of B generated by R^S_{m−1}(A,B) and
the elements b in B such that b^n in R^S_{m−1}(A,B) for some n in S .

If A is a domain, if B is the quotient field of A and S = N, then
R^S_oo(A,B) = R_oo(A) = \/{R_m(A) : 0 <= m <= oo} is the total root−closure
of A. This paper deals with the question of how many steps are needed to obtain
the total root closure of a domain. An example is given of a (quasilocal,
one−dimensional, seminormal) non−Noetherian domain A such that for all m,
R_m(A) != R_{m+1}(A) and, for each positive integer m , an example of a
(quasilocal, one−dimensional, seminormal) domain A such that R_oo(A) =
R_m(A) != R_{m−1}(A). The authors also conjecture that the total root closure
of a Noetherian domain A need not be obtainable in finitely many steps.
Reviewed by Valentina Barucci

98h:13008 13B22 (13F20 13G05)
Roitman, Moshe(IL−HAIF)
On root closure in Noetherian domains. (English. English summary)
Factorization in integral domains (Iowa City, IA, 1996), 417−−428,
Lecture Notes in Pure and Appl. Math., 189, Dekker, New York, 1997.

For a subring A of a commutative ring B and an integer n >= 1 , we say

that A is n−root closed in B if b^n in A with b in B implies b in A.
Then C(A,B) = {n | n >= 1 and A is n−root closed in B} is a multiplicative submonoid of positive integers generated by primes. In this paper, the author shows that any multiplicative submonoid of positive integers generated by primes may be realized as C(A,K) for a Noetherian domain A with quotient field K. This answers a question raised by the reviewer [Glasgow Math. J. 31 (1989), no. 1, 127−−130; MR 90b:13008]. For commutative rings A < B, the root closure of A in B may be constructed as \/ A_n, where A_0 = A and A_{n+1} = A_n[{b in B| b^m in A_n for some m>=1}]. He shows that for each positive integer d , there is a d−dimensional Noetherian domain A such that each A_n < A_{n+1} (in this case B is the quotient field of A).
This answers a question raised by D. E. Dobbs, the author, and the reviewer in [Ann. Sci. Univ. Clermont−Ferrand II Math. No. 26 (1990), 1−11; MR 92f:13006].
He also gives a simple proof of the Brewer−Costa−McCrimmon result [J. W. Brewer, D. L. Costa, and K. McCrimmon, J. Algebra 58 (1979), no. 1, 217−−226; MR 80e:13002] that C(A[X],B[X]) = C(A,B).
Reviewed by David F. Anderson
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