The Origin of a Sequence

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THE ORIGIN OF A SEQUENCE

I don't know what I can say about the length of this, except express embarrassment, for I don't believe it is any longer than it needs to be. By way of making it seem more palatable, it divides reasonably naturally into two, roughly equal parts, which I have subheaded:
Natural Number as Sequence, and
Natural Number as Set.
The first part is the largely constructive part, concerned with understanding sequence in general and the natural number sequence in particular. The second part is mainly critical, working out some of the implications of the first part for the understanding of natural number as set.

NATURAL NUMBER AS SEQUENCE

In mathematics, so far as I can tell, a sequence is always numerically ordered. In the jargon, it is a function from the natural numbers to a set. (I know this notion can be generalised in various ways, but this formulation will suit my purposes. In particular, I shall only be interested in sequences with an initial element.) And it is of course true, once we have the concept of natural number, that every sequence can be so analyzed: it has a first element, a second element, and so on.
Nevertheless, this might be thought to be an unfortunate way of looking at things, since out of the two notions, that of sequence and that of natural number, it is the former that is most primitive, and the natural numbers can be thought of as a particular example of a sequence. Counting, whilst very primitive anthropologically speaking, is not quite universal. (We are talking about an articulate number system, however underdeveloped, as opposed to a purely discriminatory number sense, which is present in various animal species.) A world without sequence, however, is unthinkable. Effective agency requires it: creating the spark then gathering the kindling does not make the fire; releasing the bow string then drawing back the bow string fails to launch the arrow. Sequence is also built into language, through its syntax.
Now I appreciate that the mathematician is not necessarily interested in giving an account of number which follows some preconceived idea of conceptual primitiveness. He has other games to play. There is

nothing necessarily wrong with analyzing a sequence as a function from N to a set. My response to this is twofold. Firstly, I think it is worthwhile attempt some conceptual clarification of the notion of sequence, not just for its own sake, but because some important insights about natural number will thereby be gained. Secondly, whilst I respect the mathematician's right to roam, so to speak, I think we will find that he very often is committed to an understanding of natural number that is at odds with these insights. For now, as a way of introducing the discussion, with the hope of engaging your interest, I would like to suggest that there is something puzzling about the situation that the natural numbers are universally applicable to any sequence (of the sort I am interested in), and yet at the same time seem just to be a particular example of sequence

apple − plum − banana

What does this sequence consist of? If we put it more properly as an English phrase:
Apple then plum then banana,
what does the 'then' mean here? Suppose we suggest some contexts: an arrangement of fruit on a table; some layered dessert concoction. Are these superfluous illustrations of sequence, or are they a kind of explanation by example of what apple − plum − banana can mean? In other words, is the meaningfulness of apple − plum − banana the possibility of interpreting it in terms of some familiar kind of order, such as a spatial one, as here?
Let us consider a deliberately absurd sequence, say:

this cup of ovaltine I am now drinking THEN philately THEN the 1963 League Division 4 football match at home for Doncaster Rovers against Grimsby Town.

One can always think of a connecting thread. Perhaps they are the three things I am going to talk about in my 2009 address to the Royal Society for the Promotion of Boredom. But then the implied sequence is not between this cup, philately and that football match, but between my talking about cup, philately and football match, in a temporal succession, or a spatial succession down the page in the order of my notes for the talk.
What would it be for the 'things' themselves, cup, philately and football match, to be in sequence? Isn't the sequence absurd, in effect meaningless, to the extent that it's impossible to provide a natural medium in which those 'things' are ordered. It has the syntactical form of a sequence, but it seems like an empty shell, a transport with no payload.
But there is one vital thing this way of looking at sequence leaves out of account. We do not merely receive order; we impose it. We are not merely perceivers of order, but creators of order. We receive order from the world in various familiar ways, spatial, temporal, alphabetical, rank in some hierarchical system, relative magnitude in some quantity or other, and so on, so that sequence, a way of taking things, is already suggested to us; nevertheless, we would like to say, we can take things any god−damn way we please. We would insist, I think, on our freedom to take absolutely anything as constituents of a sequence, and more particularly, our liberty

to combine them in artificial or spontaneous order. This kind of order is abstract, in that it is remote from all of the naturally occurring kinds of order. But it is not that we have, by it, abstracted away from all the particular kinds of order, but that this purely abstract (spontaneous, artificial) order is one of the kinds of order there are. It is not a Platonic template for all the kinds of order there are, but it does have a kind of universal applicability. For we can take any naturally occurring order, such as a temporal sequence (X happened then Y happened then Z happened) and treat it, for mathematical purposes, as merely an example of an abstract order (X then Y then Z). We can treat all order as abstract order, not because it is so, but because this defines the mathematical approach to sequence. I do not imagine that there is anything contentious about this. It is more by way of justification for something we would normally take for granted. And so far this is entirely consonant with the numeric understanding of sequence. According to that understanding, that an element is first or second is just a label that is attached to that element. The element does not have to be first or second in some respect. It is just first, or second. Indeed the natural number sequence is a universal labeller, applicable to any sequence, as already noted. What I shall be concerned with is how it acquired this peculiar status, and what this means for our understanding of the natural numbers, and of sequence.

So we think of a sequence as a succession of elements, in an entirely abstract sense. The key notion here then is an order of precedence. So that we can say:

a Precedes b (or in the passive construction, a is Succeeded by b).

We do not intend this to mean the same as 'a immediately precedes b', i.e. that a is /the/ predecessor of b. For this we need another notion, which is that of contiguity. So a is the predecessor of b if a Precedes b and a and b are contiguous, or adjacent elements. Besides precedence and contiguity, we are, as already indicated, going to give ourselves an initial element.

We naturally think of a sequence as a string or linear array. But clearly this notion is not independent of the others. It is the order of precedence that strings the elements together; the fact that every element is connected to every other by this order of precedence. More specifically, the fact that for any pair of elements, x, y,:

Either x Precedes y, or y Precedes x, but not both.

This might suggest that we could analyze the notion of sequence purely formally, in something like the following way.

Take a finite set, S = {f, e, a, g, c, b, d}, for example.
Suppose we have a binary predicate O. (Since that predicate takes an ordered pair of arguments, we are already using a notion of order. We can take that to be given by the Kuratowski set, for now, but I'll come

back to this.)
We suppose a collection of O−statements to obtain for this set, such as:

aOb, aOc, aOd, ...
bOc, bOd, ...
..
..
..
fOg

We can ask what conditions have to be satisfied by these statements, for a unique sequence or string to be produced and to justify thinking of O as the basis for that sequence. These might be:

1) For all pairs of elements x, y, either xOy or yOx but not both.
(Note that this has as a consequence not(xOx) by substitution y = x)

2) Whenever, for any x, y, z, xOy and yOz, it is also found that xOz

Notation: Let pOqOrOsO....... mean pOq and qOr and rOs and etc..

Under those conditions, there will be a uniquely derivable statement, e.g.

aObOcOdOeOfOg

where each element occurs exactly once.

There will be two distinguishable 'ends':
a unique element 'a' such that for no x is it the case that xOa
and a unique element 'g' such that for no y is it the case that gOy.

Then, arguably, O is an order notion, forming the spine, so to speak, of the sequence:
a, b, c, d, e, f, g.

The trouble is that there are two, mutually incompatible order notions which satisfy those conditions, namely PRECEDES and SUCCEEDS. I.e.:

a precedes b which precedes c .... which precedes g,
and,
a succeeds b which succeeds c ....which succeeds g.

Each of these has an alternative expression. Taking the second one above, this could have been expressed as:

g precedes f which precedes e .... which precedes a.

We should naturally call this sequence the inverse of the original sequence, where it is understood that a sequence and its inverse are

identical, or at least equivalent sequences. We obtain the inverse by inverting the order notion and reversing the sequence (i.e. switching the start and end points). There are many natural sequences which clearly demonstrate this. For example, the ranks of the playing cards, Ace, King, Queen, Jack. This is:
AKQJ in descending order, or
JQKA in ascending order.
It is as if the order notion in a sequence comes with a direction indicator, which also determines from which end the sequence is taken to start.

It is true that we can think of any sequence, abstractly, as a succession of elements, even the sequence:

a succeeds b which succeeds c which ...... succeeds g.

But we must allow that any succession has its inverse, and the inverse of this particular 'succession' is:

g precedes f which precedes e which .... precedes a.

A finite sequence comes as a pair of variants, inverses of each other and equivalent, neither of which need be thought of as primary or canonical.

The O predicate is ambiguous. And in the 'sequence' it produces, though the ends are distinguishable from one another, there are no grounds for preferring one as the start and the other as the finish.

That the Kuratowski set has its inverse, { {b}, {a, b} }, which would do equally well what { {a}, {a, b} } does, mirrors perfectly this ambiguity, in this limiting case of the sequence, the ordered pair.
The set { {a}, {a, b} }, under the Kuratowski version of events, = (a, b)
The very same set, under the inverse Kuratowski variant, = (b, a)
Clearly, choosing one or the other, the Kuratowski set or its inverse variant, is equivalent to insisting that, from an unordered pair [a, b], that 'a' be first and 'b' succeed it, or that 'b' start and 'a' succeed it, respectively.

There are I believe more serious obstacles to thinking of a sequence as fundamentally a set, that is, as a set together with some kind of sorting constraint. In the example above we gave the suggestion that the sequence was going to come out in the alphabetical order: a, b, c, d, e, f, g. Of course the same kind of constraints would equally gladly produce any sequence of letters (or would if it worked), depending on the actual content of the O−statements. It is easy to believe that the usual alphabetical order is just a preferred sequence, an arbitrary convention. (Notice though that we introduced the set as a sequence, namely the sequence f−e−a−g−c−b−d, where of course we understood the order was arbitrary. Should we have put the letters in a bag, like so many Scrabble tiles, so as to avoid at least the appearance of relying on the notion of sequence? But then the collection of O−statements, partially listed, was again presented as a sequence, the sequence in which aOb preceded aOc, and that preceded aOd, etc.. Should we put those statements in a bag too? And the conditions, 1) and 2): another little sequence. Is sequence merely ubiquitous, or is it actually inescapable?) But we have already given an example in which the sequence of elements is not arbitrary, namely the card rank sequence AKQJ (descending order). Could this be different from what it is? Well, let's just consider the K, Q, J. Clearly these reflect courtly status, with the King pre−eminent, his Queen (wife) below him, and the lowly Knave (the page, or whatever he is) below them both. Then one could imagine that the history of the development of playing cards had been different, that, for instance, the courtly status had been subverted, been inverted, so that a Jack was taken to beat a Queen, and a Queen beat a King. (According to some accounts, just such political gesturing was at work in the history of the Ace, accounting for its peculiarly ambivalent high/low status in many card games today.) But that didn't happen. And given that it didn't, there is no other sequence, no other permutation of A, K, Q, and J. The cards can be shuffled, the symbols for the cards can be shuffled around the monitor display, but the ranks are inviolable if they are to reflect the nature of the card games we play. In other words, the notion of an Ace or King or Queen or Jack in cards has sequence built into it, just as is the case with the other cards but there it is done numerically. The actual ranking of the cards is not one permutation amongst others, it is part of the very practice of card−playing. Or consider the relative power or value of the chess pieces. Beginners are often given (or used to be given) a points guide for this: Queen = 9, Rook = 5, Knight or Bishop = 3, Pawn = 1. We may just extract the sequence (descending order): Q, R, N/B, P. Here it is not this ranking as such which plays a direct role in the game, but it is a necessary consequence of the rules of chess being what they are. (It is in fact a generalisation, an 'average position' ranking, but it is the correct generalisation.) Are these examples exceptional for any reason?

There is another respect in which sequence should not be thought of as an arrangement of pre−existent, ready−made material, and this concerns novelty, or invention. Consider the phrase: Eeney, Meeney, Miney, Mo; the beginning of a children's verse for making an arbitrary choice from a small number of alternatives. Here the words (nonsense−words, at least) are as newly minted as the sequence. Is neologism trivial? Or consider a whistler. He is not whistling any remembered melody, but, being in good spirits, he is simply whistling away free−style. It would presumably be far fetched to suppose that he is making a series of very rapid choices as to which note to whistle next. But in fact where is the necessity that he is whistling in any kind of Western octave tradition at all? Well, probably he is whistling in what could be broadly be described as the Western tonal tradition, in an improvisatory and exploratory way, if the West is his cultural origin, and if he is only moderately musical. And Webern's belief that postmen on their rounds would one day be whistling his atonal 'melodies' may well have been optimistic. But this is still only probability and proviso. In the remote history of music−making there would presumably have been some form of whole musical expression, before the cataloguing into component notes and beats. What tends to deceive us here, I think, is that in our everyday lives, in our actions, our speech, our musical expression, our habits of thought, we are gliding along well−worn grooves. But the possibility of something new is inherent in the whole business. The grooves cannot always have been well−worn.

So I do not believe that the notion of a sequence can be reduced to that of a set, since in various ways the existence of an element may already require or imply the existence of the sequence in which it is located. The attempt at such a reduction may help to clarify the relationship between the various aspects of the notion of a sequence, order, contiguity and so on, but I don't think a reductive explanation is possible. Is there anything else we can do instead? There is, and the best way of introducing this will be to consider the possibility of repetition in a sequence. As in, for example, the simple sequence:

A, B, A, F

In fact this possibility was something we glossed over before in trying to derive the sequence: a, b, c, d, e, f, g.
For either we were assuming that all the elements are distinct from one another, or we were taking the letters to be distinct labels for elements some of which may have been identical to each other and then we are taking it for granted that we know how to differentially label multiple identical elements. Relying on a numerical understanding of sequence, the sequence above is easily dealt with. For there is a first and a second occurrence of A. Or more accurately, since every position in the sequence has a numeric tag, we have A in the first position and A in the third position. Aside from the fact that the natural numbers are a sequence which requires accounting for as much as any other sequence, do we need numeric tags to differentiate these positions? Doesn't it seem like overkill? For all that we need is the idea that every position in a sequence is distinct from every other position. The natural numbers do in fact perfectly achieve this differentiation, but the question remains as to how they do it.

What I propose is that, instead of trying to reduce the notion of sequence to something else, rather we exhibit the abstract form of a sequence. We give ourselves an initial element, I. Then we will combine the notions of order and contiguity in the notion NEXT AFTER, which we understand as producing an element next after the element I, i.e. NI. Iterate to produce the element next after that, N(NII), which we will write simply as NNI. And continue the iteration. So the abstract form of a sequence (of the sort I am interested in) is:

I, NI, NNI, NNNI, ...

The important thing to understand here is that this is both a particular sequence and a representation of all sequence. It is that particular sequence which consists of iterating the appending of an N to the previous element to produce a new element. On one level, it's merely a sequence of concatenations of letters. But understood as a representation of all sequence, the degree of iteration expresses the fact that every position in a sequence is distinct. It's the embodiment, in a sequence, of the abstract idea of a sequence. And what we have done here is not simply to abstract away from the nature of the constituents of a sequence, and from the particular kind of order a sequence may exemplify (spatial, temporal, and so on), for since we have perfect freedom to form sequence out of any elements, that would leave us with nothing but a series of blank spaces. But instead we have incorporated or substituted into the 'blank spaces' the general order notion implicit in sequence itself, so that those order terms themselves become constituents of the representative sequence. One could think of it as a second−order sequence. As a sequence it has some particular properties. As we have seen, there is no repetition of elements in this sequence. This is not because sequences cannot have repetitions of elements, but because the absence of repetition in the second−order or representative sequence /expresses/ the idea that every position in a sequence is distinct from all others. The representative sequence is also inviolable or canonical, not because all sequences are inviolable or canonical (that is, like AKQJ in cards, as opposed to some arbitrary sequence like apple−plum−banana), but because the inviolability expresses, in the second−order sequence where order is built into the constituents, that the identity of any sequence at large is order sensitive, is dependent not just on the constituents, but the order that they appear.

Now, I am going to take this new, or once−new sequence, and do something new, or once−new, with it. Given some collection of things, such as potatoes in a heap, perhaps, there is the process of removing them one at a time, taking care that none of the removed potatoes are returned to the collection, taking care overall that the collection is neither supplemented nor depleted during this process other than by the process itself. This process may be continued until the potatoes are exhausted, but though it has this natural termination, it can be broken off at any time. I am going to use our new sequence (I, NI, NNI, ...) to record or name the stages of this process. I am going to use it not just for potatoes, but for pebbles, for flints and feathers and bird eggs and berries, for mixtures of things, for intangible things, for just about anything at all. Moreover, you are going to use the sequence in just this way, and not just you but all of us, the whole tribe. And the use of these symbols, or their vocal equivalents, is going to form part of our communication with each other. Then we find that there all sorts of practical uses for this new sequence, many of them quite unforseen. What are these symbols? They are the natural numbers.

Note that it is not necessary to know explicitly or in advance that these symbols represent the abstract and general form of a sequence expressed in a 2nd order sequence, and so on, in order to be able to deploy these symbols in the way I've described. But it is rather that my using these symbols in these different contexts, for flints and eggs as well as potatoes, implies that the sequence can be understood in that way. Neither is it necessary for this particular sequence to be used. (Indeed, this particular sequence would be quite impractical.) Any sequence can be

substituted for it, or understood in terms of it, provided it has the desired properties, that is:
(a) There are no repetitions of elements in the sequence.
(b) The sequence is taken to be canonical.
(This last condition implies more than choosing a sequence and sticking to it. Or rather, what 'sticking to it' amounts to here is that any other sequence with the same constituents is, in effect, ungrammatical, which is to say that it becomes a sequence of mere symbols rather than functional counters. Compare the way that a circular movement of a rook on a chess board must be a movement of that piece as piece of wood or plastic, and not a 'move' of the rook as chess piece as constituted by the rules of chess.)

It is not necessary to have an endlessly iterable numeral system. We could have distinct symbols or words for the early stages of this process above, like do, re, mi, fa, so, la, ti, and after that we are stuck. It's too many symbols or number words to remember. Nevertheless, such a limited system, used in the way I've described, is explicable as I−NI−NNI−..., and constitutes a natural number system.

To put it in one sentence, natural number is the embodiment in some concrete, non−repetitious and canonical sequence of the abstract notion and general form of a sequence, which applied to the counting process yields a sequential articulation of multiplicity. (Where 'counting process' is the business of removing things singly as described above for potatoes, which can be adequately formulated independently of the notion of number.)

Since the natural number sequence must be some particular though canonical sequence, we can see how it can have, away from the counting process, a purely ordinal significance. The natural number sequence can be applied to any sequence because it is the every−sequence, the representation in a sequence of all sequence. And where the order in which things appear is of independent interest, as in for example a race of some kind, we can see that it is natural in such contexts to speak of a 4th, or NNNIth, or FAth past the winning line. In the context of the counting process the cardinal implication is automatic. It is nothing more than the degree of multiplicity as this is encoded by the appropriate element of the canonical sequence. Could one say that, conceptually speaking, counting is an application of the canonical number sequence, even if, genetically speaking, it is highly likely that the canonical sequence arose because of its ability to articulate multiplicity? This is not correct, for without the application to counting there is no reason for the canonical sequence to have a numerical connotation. The alphabet, for example, would serve as a canonical sequence (for sequences with no more than 26 elements, as far as the English language is concerned). This is borne out by actual practice, for if I want to present a short list of points in a presentation, it is a matter of indifference whether I present them as:

1) .....
2) .....
3) ...
etc.

or as:

(a) ....
(b) ....
(c) .....
etc.

It is no more true to say that the latter is really numerical than it is to say that the former is really alphabetical. For both forms have exactly the same function: to label the points or entries for clarity and convenience of future reference. Any canonical sequence will work as a universal labeller. The important lesson here is that there are not two, distinct notions of number, but two aspects of a single notion of number, both of which are required to articulate the notion of a natural number.

I want to move on shortly to discuss alternative accounts of the natural number, essentially the set−theoretic conception, but there is one more point I want to make in this section This concerns different number systems, or what would I suppose more properly be called numeral systems. (I would rather have avoided the word 'numeral', if at all possible, though in fact I think I have already used it. For some mathematicians, otherwise apparently intelligent, the number/numeral distinction seems magically to effect a division between mind and matter, with the numeral consigned to the mundane physical realm, and the number elevated to lofty region of ideas. For others, the fact that a natural number given in a certain base has a different expression in another base seems to have the same bizarre effect, as if the number they both represent was then some third, other thing, and again Number is lifted up from our physical sight like the Ascending Christ. One would have thought mathematicians would be comfortable with the idea of something having both physical and functional characteristics. Very likely, it is otherwise distorted thinking about natural number, which I am coming to shortly, which contributes to this thinking.) So far we have just two:

I, NI, NNI, ...

where the letters are abbreviations for the constituent order notion, but which as symbols serve as a numerical sequence. And the rather limited:

do, re, mi, fa, so, la, ti.

And we can contrast those with the modern, denary, positional system.

It's a curious fact that the notation which best reflects the conceptual nature of number, our unary−like I−NI−NNI−, is practically useless for anything but the smallest numbers. Almost in the same breath that we are talking about the invention of number we are talking about the invention of some effective numeral system, a system that has some form of

compression or economy. Having some standard symbol for a certain multiple, like the Roman V, X, and C, is a help. And I suppose we regard our modern system as pretty nifty. It should be pointed out, though, that the opacity of a number like NNNNNNNNI is not unique to the system. It is exactly the same phenomenon with a large, modern number like:
8987654099002134665710348 9

We know it's a particular number; it's /that/ number. But it's not easy to take in. It would be difficult to compare it with a similarly large number on another page. Which is of course why we resort to standard form.

NATURAL NUMBER AS SET

The main alternative route to the notion of natural number is well known, and is based essentially on the notion of a one−to−one correspondence. A one−to−one correspondence is a state of affairs that obtains between two collections, and can be known to obtain, without anyone knowing how many there are in each collection, and without anyone having even thought of the notion of number hitherto,. Our access to this objective fact about two collections, then, must be through the process of determining whether two collections are one−to−one. This is done by pairing off the objects from the two collections until one or other or both of the collections are exhausted. (One will find in the literature examples like an orderly arrangement of knives and forks on tables, or a procession of soldiers each with rifle and helmet. In such cases there are perceptual short cuts to the pairing−off process. In general, for two arbitrary collections, there will not be any facilitating arrangement of the objects themselves. It is the /taking/ of the pairs of objects, one pair at a time, which is in sequence.) There are, as suggested, three distinguishable possibilities in comparing in this way two collections, A and B. Either A−elements and B−elements exhaust together, or one or other of A− elements or B− elements exhaust before the other. And the idea is that, without presupposing any concept of number such as I have provided above, a concept of number (cardinal number) is /engendered/ by these distinct possibilities, corresponding to the equinumerosity of A and B, or there being either more As than Bs or more Bs than As. Number, or rather (in−)equality of number, is evinced by the, as it were, tangible difference between these three possibilities.

It should be said that such a viewpoint is not merely an alternative conception to mine, but a competing one. I have talked of associating some arbitrary name or symbol (in effect a numeral) with each step in the process of removing elements from a collection one at a time. It is routine in the set theory literature to describe this as the effecting of a one−to−one correspondence between numeral and object.

Let's have a look at this process of pairing off the elements of two arbitrary collection in a step−wise fashion to determine whether or not they are equinumerous.

I have two collections, one of potatoes, one of parsnips. I take a potato, and take a parsnip, and remove them. (It would amount to the same if those objects were marked or labelled in some way, as already counted.) I remove another pair, potato and parsnip; then still another, and so on. This is the same process as I have already described for a single collection, but here we are performing it with two collections, in step with each other. (Since the symbolic representation of precisely this process, properly understood, is sufficient to generate, indeed constitute natural number, one might wonder why anyone would go to the trouble of complicating things with a dual process to arrive at a notion of (in−)equality of number. This is a matter to which I shall return.) I continue till either one or other or both collections exhaust. Now, at any arbitrary point in this process, since the potatoes and parsnips I have taken so far /could/ have been the extent of the two collections, anything that happens after that point is a red herring. If a notion of equality of number is suggested or supported for the sub−collections I have taken so far, then what happens afterwards is irrelevant, and the exhausting simultaneously or otherwise distinction is not functioning at all. If anything is suggesting equality of number here, it is the fact that the sequences of taking potatoes and of taking parsnips are in step.
What does it mean for those sequences to be in step? Is it immaterial which step we are at? Well, we have to have that every step in this process is different from every other, both those already gone and those still to come. If some step we had taken with the potatoes could be regarded as the same step as some earlier step, what would it mean any longer for the two processes, with potatoes on the one hand and parsnips on the other, to be in step? To be in step, the two 'counts', the step−wise processes of removing potatoes and parsnips, must be at exactly the same step, which step is distinct from every other in the process. And isn't it obvious by now that not only the exhausting−together/not−exhausting−together distinction is irrelevant, but also the whole dual nature of this story, and that what we are reproducing is the very account of number which I have already provided, but through a pathological double−vision. We take the initial element, the next, the next after that, and so on. It is this which suggests a concept of number. Years ago there was a tacky British TV game show at the end of which the winning contestant was faced with a line of consumer goods emerging on a conveyor belt, and he got to keep what he could afterwards remember. Suppose instead his task had been simply to select one object, and he did this by saying: the next after the next after the next after the next after the next after the next after the next after the next after the initial object. So was that the cuddly toy or the table lamp? This is the illusion that the set−theorist wishes to foster, that we are lost in this train of 'next's.
But whilst indeed we may easily lose track of where we are here, there is no question that the contestant has chosen a specific object. The very idea of those two processes with potatoes and parsnips above being in step requires our understanding that we at some distinct and unique step, even if that step we are at is no longer easy to scan. And it is this idea, the idea of a sequence as a succession of unique, distinct steps which is at the heart of natural number.

Why has this concept of natural number not been more readily forthcoming or accepted? What explains the grip of this belief in the pivotal role of the one−to−one correspondence? I think there are some fairly specific reasons for this, and if you have followed me so far, perhaps appreciating these will help to convince you.

Firstly, there is operative what might be called the illusion of the well−worn groove. How could something as flimsy as the utterance of some mantra in time with the process of removing single potatoes from a collection of potatoes amount to the invention of something as weighty as number? As if number had been sung into existence! But as we have already indicated, it is not a single invocation of the mantra but its universal adoption, the universality of contexts, its adoption in the communication and life of the tribe. One should understand that the momentum of number built up gradually. It's not difficult to imagine, with hindsight, the needs it facilitates in even a primitive or burgeoning society: the possibilities of record−keeping, of simple commercial transactions, the division of time, the possibility of measurement, and all that those things in turn imply. Until today number is ubiquitous in everyday life, and the cornerstone of science and technology. There is nothing pre−ordained about this. Nor could anyone have said at some stage, something like: "I need to be able to compare the lengths of things even when they are not to hand −− I'll invent number for the purpose." Anymore than our ancestors could have sat down together to invent an alphabet in order to be able to form words and speak. Number has succeeded, like a speculative business enterprise, contingently, because it meets a need or finds a responsive market, because of luck, because, for whatever reason it catches on, and eventually we find it difficult to imagine life without it. It has the permanence, not of a building erected on solid foundations, but of a woodland path or track. How conveniently the bush and undergrowth parts, as if, like the Red Sea, its division had been ordained. But the path is only created and maintained by our constant traversing of this way. How solid number sees. But it is all from something that might have been as trivial, indeed once was as trivial, as Eeney, Meeney, Miney, Mo.

There is a philosophical position that our concepts reflect the objective features of the world, that those features and our concepts (when properly parsed) have, as it were, the same logical shape. It's a point of view most elegantly by Wittgenstein's Tractatus. But it's old and tired and wrong. Language is not like that. This is not how we are situated in the world. What is in common between three pebbles and three sheep is not some complex state of affairs involving relations between collections of objects, nor anything at all like that. Until number is invented there is nothing (relevantly) in common. For what is in common is, precisely, that there are three. So is number based on nothing at all? Isn't there something about the world which at least suggests or prompts the invention of number? One can point to some possibly relevant, contingent facts. The sheer repetitiousness of nature. The iterative character of biological reproduction. Perhaps the periodic changes associated with sun, moon and stars played their part. (Though strictly speaking what this last suggest is cyclic rather than linear sequence as such.) But then the relationship between number and the world cannot be like that between a picture and what it represents, nor like that between a verbal expression and the thought it represents. It is something quite different, and altogether looser. For we are talking about the motivation for a concept. Perhaps just that is the best analogy we can come up: that the relationship between number and the world (the world as apprehended and perceived by us but without yet number in it) is like the relationship between an action and its motives, motives presumed or speculated about, after the deed is done.

A second aspect of this concerns set theory and, ultimately, the concept of the real number. But let us go back to our idea of the Naturals as sequence. Is it even possible to extract an infinite set from the sequence of natural numbers? The usual line here is that it is clear enough what belongs to such a set, and what doesn't. Then allow that 1 belongs to it, then so must 2, and 3, and so on. Step on that elevator, and you get whisked all the way up. You've got all of them. Of course, if one wishes, one may speak of the set whose membership is given by the sequence of natural numbers. Certainly there are occasions when one wishes to speak of 'all the natural numbers', or 'all the odd numbers', and such like. But this formal possibility cannot annul the fact that the natural numbers are fundamentally sequence. One simply ends up creating two kinds of set, those which, whether finite or infinite, are closed or static, and those whose membership is determined by this dynamic or open−ended process which is sequence. N can only be a peculiar kind of set. Crucially one cannot say of it, as with other sets, that it has an infinite number of members. The entire justification for that was the idea that one−to−one correspondence provides an independent intuition of cardinal number, but as we have seen that was a mirage. There is no handle on how many of a thing there are than the natural numbers themselves, at least for finite collections. So when would we regard something as infinite in number? We take the natural numbers to be open−ended sequence. Likewise we take the rationals, which are merely the inverses of the naturals, together with their multiples, as open−ended sequence. (And as is well known, all the rationals can be arrange in a single open−ended sequence.) So how does something actually become infinite? Let us take the points on a unit line segment for example. The fractions provide a systematic way of choosing a point, which by density of the line segment, must exist between any two points. So between 0 and 1 we have points at 1 half, 1 and 2 thirds, 1,2 and 3 quarters, and so on. So is it that the whole set of points is infinite in number because we have picked out an infinitely large subset of the whole set? No. The 'match' between that sequence of fractions and the corresponding points is as idle as that between a hole and a screw which perfectly fits it but whose thread has been stripped. If that sequence of fractions (1/2, 1/3, 2/3, 1/4, etc.) is not infinite in number, then neither is the set of points /identified/ by those fractions. We have simply got ourselves another open−ended sequence. (Indeed that set could not be infinite in number without the fractions becoming infinitesmally small.) What we require to regard the points as infinite in number is not only that to each fraction (between 0 and 1) there is a point, but also that it is /not/ the case that to each point there is a fraction. If every point of the unit line segment was rational, so that to each point there is a rational, then to suppose there are infinitely many such points would have the absurd consequence that every point had been counted, had been labelled with its fraction, not in the hand−waving way of giving label to point and saying 'continue', but that the sequence of rationals had somehow been worked through and literally finished. In other words, the commitment to there being infinitely many points in the unit line segment implies the existence of irrational number, points on the unit line segment incommensurable with the unit length. What is infinite about the points on the unit line segment is not that we cannot finish labelling them, it is that, in a sense we cannot get started. Since the original interval is of arbitrary scale, the first division of it by a point produces two intervals either of which might be regarded as a new unit interval. Trying to enumerate the points in the unit line segment is not really like trying to count something which is infinitely numerous, it's like trying to kill a worm by repeated choppings into half, where each half is a surviving independent worm. In other words, it's the peculiar properties of the line segment (under the assumption of geometrical density) which make us regard it as having an infinite number of points. It has almost nothing to do with sequences of rationals. Sequence has to do with how we come at things, our taking up of them. If, for example, space is (to be regarded as) infinite in extent, it is not endless, except in relation to an observer and from his perspective. We should not think of this as some infinitely large sphere, whose surface we can approach but not reach because it is larger than any finite sphere. Such a space is infinite because it is everywhere, indifferent to our position and to our chopping it up into chunks. In a similar way there is no infinite bag of all the Naturals, just like a finite bag but bigger than any finite bag. The real trouble with the notion of an infinite set of naturals is that it is a hybrid one, which combines the idea of sequence and of an infinitude of things into one impossible package. If one calls the open−endedness of the naturals a kind of infinity then there are two notions of infinity, and the idea of an infinite set of naturals conflates them, and equivocates between the two.

I know there are some here who insist that there is a single mathematical notion of infinity, and that the problem with amateur thinking on the subject is that it often carries intuitions about infinity which are not intended to be carried by the mathematical notion, and he is insufficiently schooled to appreciate the difference. So let me try to be more precise about the difference between open−ended infinity and infinitude in number (at the risk of straying ever deeper into territory in which I am insufficiently schooled). Instead of assuming that we know what we mean by a phrase such as 'goes on forever', and indicating this by '.......', as in for example '.666......', let me try the opposite approach, and try to explain what could be meant by '........' in various contexts. For example:

1, 2, 3, ...

..6666......

10110 10110 10110 ....

3.141592653......

Take the first example. Only I'll write:

1, 2, 3, etc. up to and including n, ...

Then I'll say what '......' means here is: (n + 1), ...
So that: 1, 2, 3, _______ n, ...
= 1, 2, 3, _______ n, (n + 1), ...
= 1, 2, 3, _______ n, (n + 1), (n + 2), ...
etc.

In the second example, we take '......' to mean '6......'. So that:
..666........
= .6666.....
= .66666.....
etc.

Likewise in the third example, '.......' means '10110.......'

In all three cases something is repeated over and over again: a digit or a sequence of digits or in the first case the /operation/ of adding one to the last digit.

One could compare these with recursive acronyms. For example there was a make of PDA (Personal Digital Assistant) called YOPY, which stood for:

YOUR OWN PERSONAL YOPY

All good geeky fun, of course. But why not take it seriously as an analogy for an open−ended infinity? Should one believe, in say the second example, that there is an infinite string of 6s, as it were the result of all this recursion, which consists simply of 6s, all '......'s eliminated, and is literally infinite? No doubt we are used to thinking of decimal strings in this way. But is it natural to do so? Would one think that there is an infinite string consisting entirely and only of YOPs, with no trailing Y? The recursive interpretation gives us everything we need to know. Where is the necessity for the extra assumption that the '.......' is eliminable in an infinite string, when this makes no functional difference? But then what about the last example, the pi string? (Of course you will have to take it on trust that I intend the pi−string, since I can never write down enough digits to identify it conclusively as the pi−string.) There is no kind of pattern or repetition here. What can be indicated by '.......' here, but 'all the rest of the digits'? If we believe there is a unique string here, specified by its digits, then we have to believe that this is an infinitely extended, and not merely open−ended sequence, where every step of the decimal expansion contributes to the infinite amount of information it encodes. It seems like the elevator again, but with a vengeance. I agree that there is /something/

infinite about pi, or any real number (see below). But as we have already seen in the case of the points on a unit line segment, it is a fallacy to deduce that thete is an infinite number of some object or element merely from the fact that to each n or rational in an open−ended sequence of rationals there is a corresponding object or element. So here too we do not have that there is an infinite /number/ of bits of information in the pi string. It is only if we already assume that there is an infinitely extended decimal expansion of pi that progressive finite expansions reveal more and more information about /it/. On the contrary, our understanding of N as open−ended sequence allows us to say that if there is no end to the characterisation of pi by its decimal expansion, then there is no complete characterisation of pi by its decimal expansion. There is no such thing as the infinitely extended sequence, that hybrid and deformed notion of a stream which though endless is yet encompassed, as if the tap supplying it had somehow, out of sight, run dry. The calculation of the pi−string is every bit as iterative as the production of the natural number string itself. If there is an elevator here, it is one where the floors do not exist independently of its motion.

If one was sympathetic to this critique, there are perhaps still two kinds of response to it. The first is simply to dismiss the idea of an infinitely extended sequence, as doing violence to the notion of natural number and the notion of sequence itself. If I'm right in believing that all the usual ways of construing the real number (not just the idea of the real as an infinite decimal expansion) rely, in one way or another, on the idea of an infinitely extended sequence, then the notion of the real number needs to be completely recast. This is not inconceivable. I mentioned in another post some time back that Dedekind had considered the possibility of defining the real as a ratio of magnitudes in a continuous medium, though only summarily to dismiss it. The motivation for the dismissal is clear. The natural numbers or the rational numbers do not seem to require any reference to something outside themselves, so why should the real number make any reference to an outside medium. We want the numbers to be self−sufficient, to be, so to speak, their own medium. But if the notion of counting is essential to the understanding of the natural number, is it true that natural number is self−sufficient? For is it not rather the case that, at the very abstract level of 'thing', 4 is four of something, and 1/5 is a fifth of something? And if there is a reference to a world of 'things' in the notion of a natural number (which of course makes no difference to the mathematical operations we perform with these numbers), then is it not also reasonable to suppose that the concept of a real number does or should make reference to a world of continuous media in which it has its application? (So that, for example, in the context of the unit line segment, that place which is (pi − 3) is located with infinite precision, for this is a corollary of the fact that there are an infinite number of points in the line segment, but to accord infinite precision to the open−ended sequence .14159265.... is to reflect back to the representation that which is only properly attributed to that which is represented.)

The second sort of approach is to accept the oddity or artificiality of the notion of an infinitely extended sequence, but we make it the defining characteristic of the real number. It's an idealisation, where we are not concerned with fidelity to the notion of a natural number. It's as though we had some equations for gravity which required that we think of the mass of an object as concentrated at its centre of gravity. We all know the mass of an object is distributed throughout its volume, but the equations are just too good to pass up. (My physics may be even worse than my maths, if that were possible, so maybe this is just a terrible analogy, but perhaps even then it will help to make my meaning clear.) Likewise we all know there isn't a bag with all the Naturals in it, but it's too neat an idea to give up. From the point of view, so to speak, of the Naturals, the Real is a fiction, but from the point of view of the Real, the Natural is just a jumping−off point, a structure to be exploited. With the Real, we go beyond the Natural, just as mathematics has already gone beyond the Real. The Real has proved its worth, and even if it were to be entirely recast in some way, what would be produced would surely correspond to the existing understanding of the Real. In short, so the Real isn't true to the Natural, so who cares?

It's difficult to argue against a point of view like that. But even here there are consequences. For a start, we should stop offering set−theoretic accounts of natural number (and of counting) as if these were supposed to have any intrinsic cogency, as opposed to being merely means to an end. Then what of entities like 2 ^ infinity, and the idea that there are different sizes (cardinalities) of infinity? If one is engaged, legitimately enough, in the writing of a fantasy tale with knights and dragons and princesses cocooned in castle towers, it may be inappropriate to disrupt the narrative with a biology chapter on the anatomy and physiology of the dragon.

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