Riemann surface for iteration of complex map?

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Hi.

I once had a thread here where I mentioned the idea of using a Riemann surface to "continuously iterate" a complex map:

http://groups.google.com/group/sci.math/msg/3a51089884ec50d2?hl=en&dmode=source

and got hung up on how to do the "addition". So I'm wondering: what if maybe one doesn't need that −− but instead needs a different Riemann surface? What if the "best" choice of Riemann surface depends on the map we want to iterate continuously, i.e. even for the mapping z^2 + c, it would depend on c?

First of all, some introduction: consider a holomorphic function f on the complex plane, or perhaps the plane minus some isolated points (not sure exactly what types of domain should be allowed). Then consider a Riemann surface S that covers f's domain C−plane with covering map mu, and another map g: S−>S. These are all interrelated by

mu(g(s)) = f(mu(s))

and we say "g casts f under mu".

Here I call f the "shadow" of g, or g a "bulk" or "expansion" of f, and S an "iterative Riemann surface" or a "dynamical Riemann surface" of f, for lack of better terms.

Note that in this space, iteration of g corresponds to that of f, which is the property we exploit.

Now the question is, how can we find the other items for a given f−map?

For some simple elementary functions, this is not too hard. Consider the power map: f(z) = z^n, n a positive integer. When iterated, this map gives f^k(z) = z^(n^k). However, multivaluedness problems pop up for real k, and worse yet, for a single−valued "branch" we do not even have a true dynamical system!

Yet we can define a bulk for f by removing z = 0 and using the Riemann surface of the complex logarithm for the dynamical surface of f. On this surface, points are pairs s = (r, theta), where r is a positive real and theta any real. Then mu(s) = re^(itheta) is the covering map, and the bulk is g(s) = (r^n, n theta). It is easy to see that this will cast f. We can now produce iteration: g^k(s) = (r^(n^k), n^k theta), and this _is_ a dynamical system.

For the linear family, i.e. f(z) = az + b, the DRS is just the complex plane, C.

However, what about for other maps, like f(z) = z^2 − 2? This one also has a closed−form iterate f^k(z) = cos(2^k arcsin(z/2)), that of course is not dynamical for real k. But can we extend this to some sort of Riemann surface in the same manner we did for the other map, even if it is _not the same one_ (which may therefore obviate the need to "define the subtraction of 2" that I was hung up with before.)? If so, what's this Riemann surface, what is the g−map, and how do we represent points on the surface? What about the map f(z) = z^2 − 1? It has no closed−form iterate (at least not in terms of elementary functions, or any widely−used nonelementary special functions, as far as I know), but can we construct a dynamical Riemann surface for it? How about for maps z^2 + c with c−parameters from several other areas of the Mandelbrot set −− do these DRSes get more intricate and interesting forms? What about transcendental mappings, say the exponential family exp(cz), which is the basis of tetration? Does exp (z) (natural exponential), for example, have an interesting−looking DRS (DRSes?)? If so, what would the graph or graphs look like?
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