Re: Exotic Drums and the Helmholtz Equation

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Greg Egan wrote:

> Suppose we have a drum whose membrane is a connected subset M of R^2 with a piecewise smooth boundary, and we assume that the displacement of the membrane f(x,y,t) obeys a lossless, dispersionless wave equation:
>
> lap f = (1/v^2) @^2 f / @t^2
>
> where "lap" is the 2−dimensional Laplacian. The boundary condition is that f is zero on the boundary of M.
>
> If the drum has any pure tones, there will be solutions for f that separate into the form:
>
> f(x,y,t) = F(x,y) cos(omega t)
>
> and F will be a solution of the Helmholtz equation:
>
> lap F = −k^2 F
>
> where k = omega / v. F will be zero on the boundary of M.
>
> Now, I know how to find analytical solutions of the Helmholtz equation when the boundary is "nice" −− when it consists of curves of constant coordinate values, for some coordinate system in which the equation is separable. So rectangles, circles, annuli, and wedges are all easy to find solutions for.
>
> In those nice cases, there is a very simple pattern to the modes: for each coordinate, there can be any positive integer number of peaks in the standing wave across each coordinate direction. So a rectangular membrane has modes consisting of n peaks in one direction and m peaks in the other, with the lowest−frequency mode containing just a single peak, and F(x,y) is zero only on the boundary.
>
> My two questions are:
>
> (1) Is there an existence theorem which says that there *must* be

> solutions of the Helmholtz equation for an arbitarily−shaped membrane? In other words, are there always eigenfunctions of the Laplacian with real, negative eigenvalues which are zero on any given piecewise smooth simple closed curve? Or is it possible to build a drum which (even under our idealised assumptions) is physically incapable of producing a pure tone?

I think that under very general conditions on the boundary (including it being a piecewise smooth simple closed curve) the drum will have infinitely many pure tones of successively higher frequencies. I can't give the proof off hand, but I can give a sketch.

Let D be the drum surface. The solution of the Helmholtz equation with the conditions that you specified comes down to the problem of diagonalizing the Laplacian operator on the space of square integrable functions on D, L^2(D), with some caveats. The caveats come from the fact that the Laplacian is an unbounded operator (recall that taking derivatives usually makes functions more singular). Since it is not defined for all elements of L^2(D), we have to choose a dense subspace on which it is defined. The twice differentiable functions which vanish at the boundary will do nicely. Note that this choice is affected by the desired boundary conditions.

To be rigorous, one would now have to show that the Laplacian defined on this domain can be extended to an unbounded but closed self−adjoint operator on L^2(D). This part is very technical, but this is indeed possible for the Laplacian and many other elliptic differential operators. Once we have the above fact, the spectral theorem can be applied. Unfortunately, this does not preclude a continuous spectrum. But even in this case, there will be vibrational solutions which approximate pure modes to any degree of accuracy.

The last step of the proof would be to convert the Helmholz equation to an integral one. In this way, we look at the spectrum not of the Laplacian, but it's inverse, which is a bounded, even compact, operator on L^2(D). Compactness is a technical property that guarantees that the spectrum is completely discrete and has zero as the point of accumulation. A discrete spectrum guarantees the existence of a complete set of eigenvectors, both of the Laplacian and its inverse.

The inverse of the Laplacian is the integral operator whose kernel is the Green function of the Poisson equation with the same boundary conditions. Its compactness as an operator on L^2(D) follows from the Green function's regularity (smooth, bounded everywhere except a neighborhood around the point disturbance, divergence at that point no stronger, for 2D, than logarithmic). These regularity properties can be observed by placing a small massive bead on the surface of a drum and observing its deviation from the unstretched configuration. It seems reasonable that these regularity properties will be observed for any physically realizable drum. I think the last conclusion is enough to answer your original question as I did in my first paragraph above.

> (2) If/when solutions exists, must there always be one solution which is zero only on the boundary (as is certainly the case for "nice" boundaries)?

I think the answer to this question is also affirmative. One way to see it is to invoke the heat diffusion equation in place of the wave equation. Imagine the a conducting plate of the same shape as the drum, whose boundary is kept at some reference temperature (which can be taken as zero):

Lap f = @f/@t, f = 0 on the boundary.

The Helmholtz equation is still relevant here because because Laplacian eigenmodes, Lap f = −k^2 f, will correspond to temperature distributions which will be uniformly damped by diffusion, f(t) = exp(−k^2 t) f. Take the eigenmode f with the lowest allowed value of k, this is the fundamental mode. Because of exponential damping, for any initial temperature distribution, if we wait long enough, only the fundamental mode will survive (up to negligible contributions from other modes which will have decayed much more quickly). So, if we start with a uniform positive temperature distribution, as we wait, some heat will leak through the boundary and and the remnants try to distribute themselves through diffusion all around the plate. However, the temperature will remain positive everywhere except the boundary. And, if we wait long enough, the profile of the temperature distribution will be proportional (up to exponentially suppressed contributions) to the fundamental mode. From which it follows that the fundamental mode is everywhere positive (and hence non−zero), except at the boundary. I believe this property of the heat diffusion equation is referred as the maximum principle.

Hope this helps.

Igor

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